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# THE SINGULAR IDEALS AND GOLDIE DIMENSIONS OF GROUP RINGS

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## 1. Introduction

The quotient rings of group rings have been studied by many authors ([2], [3], [4], [6] and [12]). The conditions for a ring to have a self-injective semi-perfect quotient ring are well-known ([9]). In this paper, we deal with self-injective semi-perfect quotient rings of group rings.

Singular ideals, nilpotent radicals and Goldie dimensions are important to investigate quotient rings. K. Brown has proved that if $G$ is a torsion-free Abelian group then (i) $Z(A)G=Z(AG)$, where $Z(A)$ denotes the right singular ideal of the ring $A$, (ii) $N(A)G=N(AG)$ whenever $N(A)$ is nilpotent, where $N(A)$ denotes the nilpotent radical of $A$, and (iii) the right Goldie dimension of $A$ equals the right Goldie dimension of $AG$.

In this paper, we obtain the results (i) and (ii) for $G$ a free group. But the result (iii) is not true if $G$ is a free group (Example 2.5)

Throughout this paper the letter $A$ means a ring with an identity 1.

## 2. Results

A ring is called a *right finite dimensional ring* if there do not exist infinitely many nonzero right ideals whose sum is direct.

PROPOSITION 2.1. *Let $A$ be a finite dimensional ring and $G$ be a finite group. Then the group ring $AG$ is also a finite dimensional ring.*

The proof of the above statement is routine. Here we omit its proof.

PROPOSITION 2.2. (K. Brown [2]) *Let $A$ be a ring and $H \lhd G$ such that $G/H$ is a torsion-free Abelian group. Then the right Goldie dimension of $A$ equals that of $AG$.*

A group $G$ is called an *FC-group* if every element of $G$ has only finitely many conjugates.

COROLLARY 2.3. *Let $A$ be a finite dimensional ring and $G$ be an FC-group with only finitely many torsion elements. Then $AG$ is also a finite dimensional* *ring.*

PROOF. Let $T$ be the torsion subgorup of $G$. Then $AT$ is finite dimensional by Proposition 2.1. But $G/T$ is a torsion-free Abelian group ([5] ; page 676), hence $AG$ is also a finite dimensional ring by Proposition 2.2.

PROPOSITION 2.4. *Let $A$ be a right order in a self-injective semi-perfect ring $Q$ and $G$ be a finite group. Then the group ring $AG$ is also a right order in a self-injective semi-perfect ring.*

PROOF. By Proposition 2.1, $AG$ is a finite dimensional ring. Since $G$ is finite, we can see that $QG$ is also self-injective ([5] ; page 663). Thus we may assume that $E(AG)\subset QG$ (indeed, they are equal). Let $x=\sum_{i=1}^{n} b_i a_i^{-1} g_i \in E(AG)$ where $a_i$, $b_i \in A$, $a_i$ being a non-zero-divisor in $A$. Then there exists a non-zero-divisor $a^*$ in $A$, and hence a non-zero-divisor in $AG$, such that all $a_i a^* \in A$ (see [7]). Hence $xa^*=\sum_{i=1}^{n} b_i a_i^{-1} a^* g_i \in AG$. Thus we have our proposition (see [9]).

Let $G$ be a free group. Then every element of $G$ can be written as a reduced word of generators. Now we introduce the lengths of $G$ as follow:

Let $x=g_1^p g_2^q \cdots g_n^r \in G$, where $p, q$ and $r$ are nonzero integers and $g_i$ are generators such that $g_i \neq g_{i+1}$. Then we define the *length* of $x$ as $n$.

EXAMPLE 2.5. Let $G$ be a free group generated by the set $\{g_1, h_1, g_2, h_2, \cdots\}$. Take $A$ the ring of integers. Consider right ideals $(g_1+h_1)AG$, $(g_2+h_2)AG, \cdots$. We claim that their sum is direct. Indeed, suppose $(g_1+h_1)(x_1+\cdots+x_k)+\cdots+(g_n+h_n)(y_1+\cdots+y_m)=0$. Choose an element among $x_i, y_j$ so that it has maximal length among them. We may assume that such element is $x_1$. Then either $g_1x_1$ or $h_1x_1$ is of maximal length in the above sum. Thus either $g_1x_1$ or $h_1x_1$ can not vanish. It is a contradiction.

Hence $AG$ is not of finite dimensional, but $A$ is a finite dimensional ring. Thus the Proposition 2.2 is not true for $G$ a free group.

LEMMA 2.6. *Let $G$ be a free group and $x_1, \cdots, x_n$ be distinct elements in $G$. Then there exists $x\in G$ such that $(x_1x+\cdots+x_nx)(y_1+\cdots+y_m)$ has a term coefficient 1 for any distinct elements $y_1, \cdots, y_m$ in $G$, where the multiplication is taken in the group ring $AG$ over the ring $A$ of integers.*

PROOF. Suppose $x_n=(\cdots)g^t$ is of maximal length $k$ among $x_i$ and $y_m=h^s(***)$

is of maximal length $k'$ among $y_j$. Put$(\cdots)^{-1}(x_1+\cdots+x_n)=u_1+\cdots+u_{n-1}+g^t$ and $(y_1+\cdots+y_m)(***)^{-1}=v_1+\cdots+v_{m-1}+h^s$.

(case 1) For some $u_i$ (or $v_j$) has the generator $g(h)$ in the left (right, respectively) side. Then $u_i$ and $v_j$ must be of the forms $g^{t'}$ and $h^{s'}$, respectively. It is not difficult to show that for some such $i, j$, $x_iy_j$ has coefficient 1.

(case 2) No $u_i$ (and $v_j$) contains the generator $g(h)$ in the left (right, respectively).

If $g\neq h$, then $x_ny_m$-term has coefficient 1.

If $g=h$, then it is sufficient to consider the case $h^s=g^{-t}$, i.e. every element of maximal length among $x_i$ and $y_j$ have $g^t$ and $g^{-t}$ as the last part and the first part, respectively.

If $x_1$ is of length $k-1$, we consider the following three cases:

(i) if $x_1=(\cdots)\,f^{s'}$ with $f\neq g$ then $x_1g^{-t}$ is of length $k+k'-1$. It is not difficult to show that $x_iy_m$ has coefficient 1.

(ii) if $x_1=(\cdots)g^{t'}$ with $t'\neq t$ then $x_1g^{-t}$ and $x_ng^{-t}$ are of length $k-1$ and of maximal length among $x_ig^{-t}$. But the last parts of them are not equal, and hence $g^{-t}$ is the desired element.

(iii) $x_1=(\cdots)g^t$. Now we consider only the case (iii).

If $x_2$ is of length $k-2$, we consider the following three cases:

(iv) if $x_2=(\cdots)f^{s'}$ with $f\neq g$ then $x_2g^{-t}$ is of length $k-1$ and of maximal length among $x_ig^{-t}$, but the last part of $x_2g^{-t}$ is different from that of $x_ng^{-t}$ of length $k-1$. Thus $g^{-t}$ is the desired element.

(v) if $x_2=(\cdots)g^{t'}$ with $t'\neq t$, then $x_2g^{-t}$ is of length $k-2$, and $x_ng^{-t}$ is of maximal length $k-1$ among $x_ig^{-t}$. But the last parts of $x_2g^{-t}$ and $x_ng^{-t}$ are not equal. Thus we can deduce that $g^{-t}$ is the desired element.

(vi) $x_2=(\cdots)g^t$. Now we consider only the case (vi).

Continuing this process, we obtain the fact that all $x_i$ has $g^t$ as last part. Then we can easily see that our assertion is true.

LEMMA 2.7. (R. Shock [11]) *For $0\neq\alpha\in AG$ there exists $b\in A$ such that the right annihilators of the coefficients of nonzero element $\alpha b$ are equal.*

PROPOSITION 2.8. *Let $A$ be a commutative ring and $H$ be a subgroup of a* *group G such that* $H \subset C(G)$, *the center of G, and G/H is a free group. Then* $Z(AH)G = Z(AG)$.

The proof of Proposition 2.8 is similar to that of Theorem 2.6 in [11] if we apply the Lemmas 2.6 and 2.7. Here we omit its proof.

We don't know whether the commutativity in Proposition 2.8 can be removed. But if $A$ is right non-singular, the commutativity is unnecessary.

PROPOSITION 2.9. *Let G be a free group. Then A is right non-singular if and only if AG is.*

PROOF. Assume that $Z(A)=0$. Suppose $0 \neq \alpha \in Z(AG)$ and put $\alpha = a_1x_1 + \cdots + a_nx_n$, $a_i \neq 0$ and $x_i \in G$. By Lemma 2.7, there is $b \in A$ such that the right annihilators of nonzero coefficients of $\alpha b \neq 0$ are equal. Put $b = b_1z_1 + \cdots + b_kz_k$, $b_j \neq 0$. By Lemma 2.6, there is $x \in G$ such that $(z_1x + \cdots + z_kx)(y_1 + \cdots + y_m)$ has a term with coefficient 1 for any elements $y_1, \cdots, y_m$ in $G$. Since $b_1 \notin Z(A) = 0$, there is $0 \neq c \in A$ such that $r(b_1) \cap cA = 0$. We claim that $r(\alpha bx) \cap cAG = 0$. Indeed, suppose that this is false, then there is $0 \neq \beta \in cAG$ such that $(\alpha bx)\beta = 0$. Put $\beta = c_1y_1 + \cdots + c_my_m$, $0 \neq c_j \in cA$. Then $b_ic_j = 0$ for some $i, j$. Thus $b_1c_j = 0$. It is a contradiction. Hence $\alpha bx$ can not belong to $Z(AG)$, a contradiction. So $Z(AG) = 0$. The reverse implication is clear.

COROLLARY 2.10. *Let* $H < G$ *such that* $H \subset C(G)$ *and G/H is a free group. Then AH is right non-singular if and only if AG is.*

PROOF. Clear.

PROPOSITION 2.11. *Let A be a ring and G be a free group. Assume that N(A) is nilpotent. Then* $N(A)G = N(AG)$.

The proof is similar to that of Lemma 2.7 in [2]. Here we omit its proof.

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