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THE RIESZ DECOMPOSITION OF VECTOR-VALUED UNIFORM AMART FOR CONTINUOUS PARAMETER

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Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. For each $t\in R^{+}=[0,\infty)$, let $\mathscr{F}_t$ be a sub-σ-algebra of $\mathscr{F}$ which includes all of null sets. The collection $(\mathscr{F}_t)_{t\in R^+}$ of σ-algebras is assumed to be increasing (i.e., if $t\leq s$, then $\mathscr{F}_t\subset\mathscr{F}_s$ and right continuous (i.e. $\mathscr{F}_t=\bigcap_{s>t}\mathscr{F}_s$ for all $t\in R^{+}$). A simple stopping time of $(\mathscr{F}_t)_{t\in R^+}$ is a function τ: $\Omega\to[0,\infty)$ taking only finitely many values, such that $\{\tau\leq t\}\in\mathscr{F}_t$ for all t. Let T be the set of all simple stopping times; under the natural order T is a directed set filtering to the right. σ, τ and ρ denote elements of T.

Let E be a Banach space and $(X_t)_{t\in R^+}$ be a family of random variables *adapted* to $(\mathscr{F}_t)_{t\in R^+}$, i.e., X_t: $\Omega\to E$ is $\mathscr{F}_t$-strongly measurable for each $t\in R^{+}$. $E(X)$ (expectation of X) is the Bochner integral of X and $E(X|\mathscr{B})$ is a conditional expectation of X relative to a subalgebra $\mathscr{B}\subset\mathscr{F}$. For $\tau\in T$ define the random variable X_τ by $X_\tau=X_t$ on $\{\tau=t\}$ and define the σ-algebra $\mathscr{F}_\tau$ by $\mathscr{F}_\tau=\{A\in\mathscr{F}|A\cap\{\tau=t\}\in\mathscr{F}_t$ for all $t\in R^{+}\}$. The family $(X_t)_{t\in R^+}$ is called an amart [4] for $(\mathscr{F}_t)_{t\in R^+}$ iff $E\|X_t\|<\infty$ for all $t\in R^{+}$ and the net $(E(X_\tau))_{\tau\in T}$ converges to a finite limit.

A. Bellow introduced uniform amarts [3] for discrete parameter. One of the charaterizations of uniform amarts is the following: (X_n) is a uniform amart iff $\lim_{\tau_n\geq n}(E(X_{\tau_n}|\mathscr{F}_n)-X_n)=0$ in L^1_E.

Now we introduce uniform amart for continuous parameter.

DEFINITION $(X_t)_{t\in R^+}$ is *E-valued uniform amart* if whenever $\tau_n\geq s_n\uparrow\infty$ we have $\lim(E(X_{\tau_n}|\mathscr{F}_{s_n})-X_{s_n})=0$ in L^1_E, where $\tau_n\in T$, $s_n\in R^{+}$.

Edgar and Sucheston [5] have given a Riesz decomposition of vector-valued amarts for discrete parameter, and A. Bellow [3] has given a Riesz decomposition of vector-valued uniform amarts for discrete parameter. The continuous parameter version of this theorem is the main result of the present note.

PROPOSITION 1. *If $(X_t)_{t\in R^+}$ is vector-valued quasi-martingale, then $(X_t)_{t\in R^+}$ is vector-valued uniform amart.*

PROOF. Let $\tau_n \geq s_n \uparrow \infty$, where $\tau_n \in T$, $s_n \in R^+ = [0, \infty)$. Let (t_R) be the "union" of (s_n) and the set of values of all the τ_n arranged in increasing order. Then (X_{t_k}) is quasi-martingale for discrete parameter. By [3] (X_{t_k}) is uniform amart for discrete parameter. Since (τ_n) are stopping times for $(\mathscr{F}_{t_k})$, by one of the chracterizations of uniform amarts we have

$$\int \|E(X_{\tau_n}|\mathscr{F}_{s_n}) - X_{s_n}\| dp \to 0 \text{ as } n\to\infty.$$

Therefore $(X_t)_{t\in R^+}$ is vector-valued uniform amart.

THEOREM 2. (Riesz decomposition) *Let $(X_t)_{t\in R^+}$ be vector-valued uniform amart. Then $(X_t)_{t\in R^+}$ admits a unique decomposition $X_t = Y_t + Z_t$ where $(Y_t)_{t\in R^+}$ is a martingale and $\|Z_t\|_1 \longrightarrow 0$ as $t\longrightarrow\infty$. In addition $\|Z_\tau\|_1 \longrightarrow 0$ as $\tau\uparrow\infty$.*

PROOF. First we will prove that this decomposition is unique. If $Y_t + Z_t = Y'_t + Z'_t$, where Y_t, Z_t and Y'_t, Z'_t are two Riesz decomposition for X_t, then $\|Y_t - Y'_t\|_1 \leq \|Z_t\|_1 + \|Z'_t\|_1 \to 0$ as $t\to\infty$. But (Y_t), (Y'_t) being martingales implies that $(\|Y_t - Y'_t\|)$ is a real-valued submartingale. This implies that $\|Y_t - Y'_t\|_1$ is a non-decreasing function on t. Therefore $\|Y_t - Y_t'\|_1 = 0$ which gives

$$Y_t = Y_t' \text{ a.s. and } Z_t = Z_t' \text{ a.s. }.$$

Next we will show the existence of the decomposition. Let $s_1 < s_2 < s_3 < \cdots$ be any strictly increasing squence with $\lim s_n = \infty$. Let t be any fixed real number. For $s_j \geq t$, $E(X_{s_j}|\mathscr{F}_t)$ is a Cauchy sequence in L^1_E, because

$\|E(X_{s_j}|\mathscr{F}_t) - E(X_{s_i}|\mathscr{F}_t)\|_1 = \|E(E(X_{s_j}|\mathscr{F}_{s_i}) - X_{s_i})|\mathscr{F}_t\|_1 \leq \|E(X_{s_j}|\mathscr{F}_{s_i}) - X_{s_i}\|_1 \to 0.$
It follows that $W_t = \lim E(X_{s_j}|\mathscr{F}_t)$ exists almost surely and in L^1_E. The L^1-convergence of $E(X_{s_i}|\mathscr{F}_t)$ obviously implies

$$E(W_t|\mathscr{F}_s) = \lim E(E(X_{s_i}|\mathscr{F}_t)|\mathscr{F}_s) = \lim E(X_{s_i}|\mathscr{F}_s) = W_s \text{ for } s<t,$$

that is, $(W_t)_{t\in R^+}$ is a martingale. Now we show that $\|X_{s_j} - W_{s_j}\|_1 \to 0$ as $j\to\infty$. Since $\|E(X_{s_j}|\mathscr{F}_{s_i}) - X_{s_i}\|_1 \to 0$, $E(X_{s_j}|\mathscr{F}_{s_i}) \to W_{s_i}$ in L^1. Therefore $X_{s_i} \to W_{s_i}$ in L^1.

Consider the sequence 1, 2, 3, $\cdots$. Let Y_t denote the corresponding martingale such that $\|X_n - Y_n\|_1 \to 0$. We claim that $\|X_t - Y_t\|_1 \to 0$ as $t\to\infty$. Otherwise there would exist a strictly increasing sequence $t_1 < t_2 < t_3 < \cdots$ increasing to ∞

such that $\lim \|X_{t_n}-Y_{t_n}\|_1>\varepsilon$. Let (s_n) be the union of (t_n) and (n) in increasing order. Let (R_t) be a martingale such that $\|X_{s_i}-Y_{s_i}\|_1\to 0$ as $i\to\infty$. Now (Y_t-R_t) is a martingale and so $\|Y_t-R_t\|$ is a submartingale, therefore $\|Y_t-R_t\|_1$ is non-decreasing function of t. However we have

$$\|Y_n-R_n\|_1\leq\|X_n-Y_n\|_1+\|X_n-R_n\|_1\to 0$$

by the choice of the martingales (Y_t) and (R_t). Therefore $P(Y_t=R_t)=1$ for every t. This implies that $\|X_{s_i}-Y_{s_i}\|_1\to 0$ and this is against the choice of the sequence (t_i). Therefore $\|X_t-Y_t\|_1\to 0$ as $t\to\infty$. Put $Z_t=X_t-Y_t$.

In addition, let $\tau_n\in T$, $\tau_n\uparrow\infty$. The set of values of all the τ_n is an incresing sequence $s_n\uparrow\infty$. Let $U_i=Z_{s_i}$, define stopping times σ_n for $(\mathscr{F}_{s_i})$ by letting $\sigma_n=i$ on $\{\tau_n=s_i\}$. (U_i) is discrete parameter uniform amart, and (U_i) is L^1-bounded. By [3] $(\|U_i\|)$ is real-valued L^1-bounded amart. By [1] $\|Z_{\tau_n}\|_1=\|U_{\sigma_n}\|_1\to 0$.

Metivier and Pallaumail [6] proved Riesz decomposition for quasi-martingale under the assumption that E has the Radon-Nikondym property. But combining theorem 2 with proposition 1, we obtain Riesz decomposition for quasi-martingale without Radon-Nikodym property.

COROLLARY 3. *Vector-valued quasi-martingale for continuous parameter has Riesz decomposition.*

Let now $\mathscr{B}$ be an algebra of subsets of Ω. If μ: $\mathscr{B}\to E$ is a finitely additive set function, we denote by $\|\mu\|$ total variation of μ, that is, $\|\mu\|=\sup_i \sum\|\mu(A_i)\|$ (the supremum being taken over all finite sequences (A_i) of disjoint sets in $\mathscr{B}$) whenever the supremum is finite.

LEMMA 4. [4] *Let E be a Banach space. Let $(X_t)_{t\in R^+}$ be an E-valued amart. For each $\tau\in T$*

$$\text{set } \mu_\tau(A)=\int_A X_\tau dp \text{ for } A\in\mathscr{F}_\tau$$

Then the family $(\mu_\tau(A))_{\tau\in T}$ converges to a limit $\mu(A)$ in E for each $A\in\bigcup_{t\in R^+}\mathscr{F}_t = \bigcup_{\tau\in T}\mathscr{F}_\tau$, and the convergence is "uniform" in the sense that for each $\varepsilon>0$ there is $t_0\in R^+$ such that

$$\sigma\in T,\ \sigma\geq t_0\Rightarrow \sup_{A\in\mathscr{F}_\sigma}\|\mu_\sigma(A)-\mu(A)\|\leq\varepsilon.$$

THEOREM 5. *Let $(X_t)_{t\in R^+}$ be vector-valued amart. The following are equivalent under the notations in Lemma 4.*

(1) *Given* $\varepsilon>0$, *there exists* $t_0 \in R+$ *such that* $\|\mu_\sigma-\mu|\mathscr{F}_\sigma\|<\varepsilon$ *for any* $\sigma\geq t_0$.

(2) $(X_t)_{t\in R^+}$ *is vector-valued uniform amart.*

PROOF. (1) $\Rightarrow$ (2) First we observe that $\|\mu_\tau\|=\int\|X_\tau\|dp$. Let $\tau_n\geq s_n\uparrow\infty$.

$\|E(X_{\tau_n}|\mathscr{F}_{s_n})-X_{s_n}\|_1\leq\|X_{\tau_n}-X_{s_n}\|_1=\|\mu_{\tau_n}-\mu_{s_n}\|\leq\|\mu_{\tau_n}-\mu\|+\|\mu-\mu_{s_n}\|\to0$ as $n\to\infty$.

(2) $\Rightarrow$ (1). By Theorem 1 we have Riesz decomposition $X_t=Y_t+Z_t$ where (Y_t) is a martingale and $\|Z_\tau\|_1\to0$ as $\tau\uparrow\infty$. Since $\mu_\tau(A)=\int_A X_\tau\,dp=\int_A Y_\tau\,dp+\int_A Z_\tau dp$ and $\int_A Z_\tau\,dp\to0$. Therefore $\mu(A)=\int_A Y_\tau\,dp$ is independent on τ. Hence $\|\mu_\tau-\mu|\mathscr{F}_\tau\|=\int\|X_\tau-Y_\tau\|dp=\|Z_\tau\|_1\to0$.

COROLLARY 6. *Let* $(X_t)_{t\in R^+}$ *be a vector-valued uniform amart which is* L^1-*bounded, that is,* $\sup\int\|X_t\|dp=A<\infty$.
Then $(\|X_t\|)_{t\in R^+}$ *is a real-valued* L^1*-bounded amart at* ∞.

PROOF. By theorem 2 there exists $t_0\in R^+$ such that $\|\mu_\sigma-\mu|\mathscr{F}_\sigma\|<\varepsilon$ for any $\sigma\geq t_0$. In particular for all $t\geq t_0$,

$\|\mu_t-\mu|\mathscr{F}_t\|<\varepsilon$. Therefore $\|\mu|\mathscr{F}_t\|\leq\|\mu_t\|+\varepsilon\leq A+\varepsilon$.

Since the total variation of a measure increases with the σ-field, we have

$$\|\mu|\mathscr{F}_\sigma\|\leq A+\varepsilon \text{ for all } \sigma\in T.$$

Hence $\|\mu|\mathscr{F}_\sigma\|$ converges in R. On the other hand

$$\left|\int\|X_\sigma\|dp-\|\mu|\mathscr{F}_\sigma\|\right|=\left|\|\mu_\sigma\|-\|\mu|\mathscr{F}_\sigma\|\right|\leq\|\mu_\sigma-\mu|\mathscr{F}_\sigma\|<\varepsilon.$$

Therefore $(\int\|X_\sigma\|dp)_{\sigma\in T}$ converges.

COROLLARY 7. *Let* E *has Randon-Nikodym property, and* $(X_t)_{t\in R^+}$ *be* L^1-*bounded uniform amart.*

(1) *If* $(X_t)_{t\in R^+}$ *is teminally uniformly integrable, that is, given* $\varepsilon>0$ *there are* t_0, δ_0 *such that* $t\geq t_0$ *and* $P(A)<\delta$ *implies* $\int_A\|X_t\|\,dp<\varepsilon$, *then* $(X_t)_{t\in R^+}$ *converges in* L^1-*norm.*

(2) *if* $(X_t)_{t\in R^+}$ *is separable, then* $(X_t)_{t\in R^+}$ *converges a.s. as* $t\to\infty$.

PROOF. (1) By theorem 1 we have Riesz decomposition $X_t=Y_t+Z_t$. Since (Z_t) is terminally uniformly integrable, $Y_t=X_t-Z_t$ is also terminally uniformly integrable. By [9] (Y_t) converges in L^1. Hence (X_t) converges in L^1.

(2) Let $X_t=Y_t+Z_t$ be the Riesz decomposition. Now $\int\|Z_t\|dp\to 0$, so $(Z_t)_{t>t_0}$ is L^1-bounded for some t_0. Thus $(Y_t)_{t>t_0}$ is an L^1-bounded separable martingale. Hence (Y_t) converges a.s. as $t\to\infty$. On the other hand $(\|Z_t\|)_{t>t_0}$ is a real-valued L^1-bounded amart at ∞ by Cor 2. So $(\|Z_t\|)$ converges a.s. as $t\to\infty$. Hence (Z_t) converges a.s. Therefore (X_t) converges *a.s.*

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