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RIEMANNIAN MANIFOLDS ADMITTING AN INFINITESIMAL CONFORMAL TRANSFORMATION II

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1. Introduction

In the present paper we have obtained some conditions for a Riemannian manifold M to be isometric to a sphere. For, we have applied several results of well known authors.

Let M be a Riemannian manifold of dimension n with metric tensor g_{ji}. We denote by ∇_i $K_{kji}{}^h$, K_{ji} and K the operator of covariant differentiation with respect to g_{ji}, the curvature tensor, the Ricci tensor and the scalar curvature of M respectively. We put

(1.1) $$G_{ji}=K_{ji}-\frac{1}{n}K\ g_{ji}$$

(1.2) $$Z_{kjih}=K_{kjih}-\frac{K}{n(n-1)}\ \{g_{hk}\ g_{ji}-g_{jh}\ g_{ki}\}.$$

Then we have

(1.3) $$G_{ji}\ g^{ji}=0,\ Z_{aji}{}^a=G_{ji},$$

(1.4) $$G_{ji}\ G^{ji}=K_{ji}\ K^{ji}-\frac{K^2}{n}$$

and

(1.5) $$Z_{hkji}\,Z^{hkji}=K_{hkji}\ K^{hkji}-\frac{2K^2}{n(n-1)}$$

When M admits an infinitesimal transformation v^h, we denote by $\mathfrak{L}$ the operator of Lie derivation with respect to v^h. Thus, if M admits an infinitesimal conformal transformation v^h, we have

(1.6) (a) $\mathfrak{L}g_{ji}=2\rho g_{ji}$,

(b) $\mathfrak{L}g^{ih}=-2\rho g^{ih}$,

for a certain scalar field ρ. Let $\rho_i=\nabla_i\rho$.

For an infinitesimal conformal transformation v^h in M, we have

(1.7) $$\mathfrak{L}K_{kji}{}^h=-\delta_k^h\ \nabla_j\ \rho_i+\delta_j^h\ \nabla_k\ \rho_i-\nabla_k\ \rho^h g_{ji}+\nabla_j\ \rho^h\ g_{ki}$$

(1.8) $$\pounds K_{ji}=-(n-2)\nabla_j\rho_i-\Delta\rho\, g_{ji},$$

(1.9) $$\pounds K=-2(n-1)\Delta\rho-2\rho K,$$

where

(1.10) $$\Delta\rho=g^{ji}\nabla_j\nabla_i\rho.$$

Thus in M with K=const., we have

$$\Delta\rho=-\frac{K}{n-1}\rho.$$

We have

(1.11) $$\pounds G_{ji}=-(n-2)(\nabla_j\rho_i-\frac{1}{n}\Delta\rho\, g_{ji}),$$

$$\pounds Z_{kji}{}^h=-\delta_k^h\nabla_j\rho_i+\delta_j^h\nabla_k\rho-\nabla_k\rho^h g_{ji}+\nabla_j\rho^h g_{ki}+\frac{2}{n}\Delta\rho(\delta_k^h g_{ji}-\delta_j^h g_{ki}).$$

Let us now state some wellknown results:

THEOREM A (Yano, [4]). *If M is compact orienable and of dimension $n>2$, with K=const., and admits an infinitesimal non-isometric conformal transformation v^h: $\pounds g_{ji}=2\rho g_{ji}$, $\rho\neq$ const; such that*

$$\int_M G_{ji}\rho^j\rho^i\, dv\geq 0$$

dv being the volume element of M, then M is isometric to a sphere.

THEOREM B 2. *If a compact orientable M of dimension $n>2$ with K=const. admits an infinitesimal non-homothetic conformal transformation v^h: $\pounds g_{ji}=2\rho g_{ji}$, $\rho\neq$const; such that*

$$\pounds(G^{ji}\pounds G_{ji})\leq 0,$$

then M is isometric to a sphere.

THEOREM C. *If a compact orientable M admits an infinitesimal conformal transformation v^h: $\pounds g_{ji}=2\rho g_{ji}$ then we have*

(1.12) $$\int_M \rho F\, dv=-\frac{1}{n}\int_M \pounds F\, dv$$

for any function F.

We also need following integral formulas proved in (Yano, K. [3]):

If a compact orientable Riemannian manifold M of dimension $n>2$ with $K=$

const. admits an infinitesimal nonhomothetic conformal transformation v^h: $\pounds g_{ji} = 2\rho\, g_{ji}$, $\rho \neq$const., then we have

$$(1.13) \qquad \int_M G_{ji}\rho^j \rho^i dv = \frac{1}{n-2}\int_M \left[2\rho^2 G_{ji}G^{ji} + \frac{1}{2}\rho\pounds(G_{ji}G^{ji})\right] dv$$

$$(1.14) \qquad \int_M G_{ji}\rho^j \rho^i dv = \int_M \left[\frac{1}{2}\rho^2 Z_{kjih}Z^{kjih} + \frac{1}{3}\rho\pounds(Z_{kjih}Z^{kjih})\right] dv$$

2. Some Theorems

Let us prove the following lemma first.

LEMMA 2.1. *If a compact orientable Riemannian manifold M of dimension $n>2$ with K=const. admits an infinitesimal non-homothetic conformal transformation v^h: $\pounds g_{ji} = 2\rho\, g_{ji}$, $\rho \neq$const., then we have*

$$(2.1) \qquad \int_M G_{ji}\rho^j \rho^i dv = \frac{1}{\alpha}\int_M \left[\frac{1}{2}\rho^2 W_{kjih}W^{kjih} + \frac{1}{8}\rho\pounds(W_{kjih}W^{kjih})\right] dv$$

where

$$\alpha = \frac{n}{n-1}$$

and

$$W_{kjih} = K_{kjih} - \frac{1}{n-1}[K_{kh}g_{ji} - K_{jh}g_{ki}]$$

is projective curvature tensor, representing the derivation of the manifold from projective flatness..

PROOF. A relation between Z_{kjih} and W_{kjih} is given

$$W_{hkji} = Z_{hkji} + \frac{1}{n-1}[g_{hj}G_{ki} - g_{kj}G_{hi}].$$

Then we have

$$(2.2) \qquad W_{hkji}W^{hkji} = Z_{hkji}Z^{hkji} - \frac{2}{n-1}G_{ki}G^{ki}$$

From (1.14) and (2.2) we have

$$\int_M G_{ji}\rho^j \rho^i dv = \int_M \left[\frac{1}{2}\rho^2\left\{W_{hkji}W^{hkji} + \frac{2}{n-1}G_{ki}G^{ki}\right\} + \frac{1}{8}\rho\left\{\pounds W_{hkji}W^{hkji} + \frac{2}{n-1}\pounds G_{ki}G^{ki}\right\}\right] dv$$

$$\int_M G_{ji}\rho^j \rho^i dv = \int_M \left[\frac{1}{2}\rho^2 W_{hkji}W^{hkji} + \frac{1}{8}\rho\pounds(W_{hkji}W^{hkji})\right] dv + \frac{1}{n-1}\int_M \left[\rho^2 G_{ki}G^{ki} + \frac{1}{4}\rho\pounds G_{ki}G^{ki}\right] dv$$

From (1.13) we get

$$\int_M G_{ji}\rho^j\rho^i dv=\int_M\left[\frac{1}{2}\rho^2 W_{hkji}W^{hkji}+\frac{1}{8}\rho\pounds(W_{hkji}W^{hkji})\right]dv$$
$$+\frac{1}{n-1}\frac{n-2}{2}\int_M G_{ji}\rho^j\rho^i dv$$
$$\left(1-\frac{n-2}{2(n-1)}\right)\int_M G_{ji}\rho^j\rho^i dv=\int_M\left[\frac{1}{2}\rho^2 W_{hkji}W^{hkji}\right.$$
$$\left.+\frac{1}{8}\rho\pounds(W_{hkji}W^{hkji})\right]dv$$
$$\int_M G_{ji}\rho^j\rho^i dv=\frac{(n-1)}{n}\int_M\left[\rho^2 W_{hkji}W^{hkji}+\frac{1}{4}\rho\pounds(W_{hkji}W^{hkji})\right]dv$$

hence proved.

THEOREM 2.1. *Suppose that a compact Riemannian manifold M of dimension $n>2$ with K=const. admits an infinitesimal nonhomothetic conformal transformation v^h.*

$$\text{If } \pounds\pounds(W_{kjih}W^{kjih})\leqq 0$$

then M is isometric to a sphere.

PROOF. Using Lemma (2.1) and Theorem C we have

$$2\int_M G_{ji}\rho^j\rho^i dv=\frac{1}{\alpha}\int_M\rho^2 W_{kjih}W^{kjih}dv$$
$$+\frac{n-1}{4n}\int_M\rho\pounds(W_{kjih}W^{kjih})dv$$
$$=\frac{1}{\alpha}\int_M\rho^2 W_{kjih}W^{kjih}dv-\frac{n-1}{4n}\frac{1}{n}\int_M\pounds\pounds(W_{kjih}W^{kjih})dv.$$

Now, since $\alpha>0$, as $n>2$, we can easily see in the light of theorem A, that if

$$\pounds\pounds(W_{kjih}W^{kjih})\leqq 0.$$

then M is isometric to a sphere.

TNEOREM 2.2. (for instance see Hiramatu [1], 74). *If a compact orientable Riemannian manifold M with constant scalar curvature field K and of dimension n admits an infinitesimal conformal transformation v^h: $\pounds g_{ji}=2g_{ji}$, $\neq$ const, such that*

$$\int_M\rho(\nabla^j\rho^i)G_{ji}dv\leqq 0$$

then M is isometric to a sphere.

PROOF. By using Green's theorem, we have

$$\int_M \nabla^j(G_{ji}\,\rho^i\rho)dv=\int_M (\nabla^j G_{ji})\,\rho^i\,\rho\, dv + \int_M G_{ji}(\nabla^j \rho^i)\rho\, dv+\int_M G_{ji}\,\rho^j\rho^i\, dv=0$$

Since $\nabla^j G_{ji}=0$, we have

$$\int_M \rho(\nabla^j \rho^i)\, G_{ji}\, dv+\int_M G_{ji}\,\rho^j\rho^i\, dv=0. \tag{2.3}$$

Thus the result follows from Theorem *A*.

THEOREM 2.3. *If M is compact orientable and of dimension* $n>2$ $(n>4)$ *with* $K=const.$ *and admits an infinitesimal nonhomothetic conformal transformation* v^h: $\pounds g_{ji}=2\rho g_{ji}$, *such that*

$$\int_M \rho^2 G^{ji} G_{ji}\, dv\leq 0,\ \left(\int_M \rho^2 G_{ji}\, G^{ji}\, dv\geqq 0\right) \tag{2.4}$$

then M is isometric to a sphere.

PROOF. In order to prove it, let us take the following tensor.

$$W'_{kjih}=aZ_{kjih}+\frac{b}{n-2}(g_{kh}\, G_{ji}-g_{jh}\, G_{ki}+G_{kh}\, g_{ji}-g_{jh}\, g_{ki}), \tag{2.5}$$

a and *b* being constants (Yano & Sawaki [3]). In general Yano and Sawaki have proved the following

$$W'_{kjih}\, W'^{kjih}=a^2\, Z_{kjih}\, Z^{kjih}+\frac{4(2a+b)b}{n-2}\, G_{ji}\, G^{ji}, \tag{2.6}$$

$$(\pounds W'_{kjih})\, W'^{kjih}=2\rho W'_{kjih}\, W'^{kjih}-4(a+b)^2 G_{ji}\,\nabla^j \rho^i, \tag{2.7}$$

$$(\pounds W'_{kjih}\, W'^{kjih})=-4\rho\, W'_{kjih}\, W'^{kjih}-8(a+b)^2\, G_{ji}\,\nabla^j \rho^i \tag{2.8}$$

Using (2.6)–(2.8) in (1.14) we get

$$\int_M G_{ji}\,\rho^j\rho^i dv=\int_M \Big[\frac{1}{2}\rho^2\Big\{\frac{1}{a^2}\, W'_{kjih}\, W'^{kjih}-\frac{4(2a+b)b}{a^2(n-2)}\, G_{ji}\, G^{ji}\Big\} + \frac{1}{8}\,\rho\,\Big\{\frac{1}{a^2}\pounds(W'_{kjih}\, W'^{kjih})-\frac{4(2a+b)b}{(n-2)a^2}\pounds(G_{ji}\, G^{ji})\Big\}\Big]\, dv$$

$$=\int_M \frac{2b(2a+b)\rho^2}{n-2}\Big(\frac{n-4}{4a^2}\Big)\, G_{ji}\, G^{ji}\, dv + \frac{-(a+b)^2}{a^2}\int_M \rho\, G_{ji}\,\nabla^j \rho^i\, dv$$

Using (2.3), we get

$$\frac{a^2-(a+b)^2}{a^2}\int_M G_{ji}\,\rho^j\,\rho^i\, dv$$

$$=\frac{b(2a+b)(n-4)}{2a^2(n-2)}\int_M \rho^2 G_{ji} G^{ji}\, dv$$

$$\int_M G_{ji}\,\rho^j \rho^i\, dv=\frac{4-n}{n-2}\int_M \rho^2 G_{ji} G^{ji}\, dv$$

hence we get the result for $n>2$.

THEOREM 2.4. *If a compact orientable M with $K=const.$ of dimension $n>2$ admits an infinitesimal conformal transformation v^h: $\pounds g_{ji}=2\rho g_{ji}$, such that ρ does not vanish on any n-dimensional domain and*

(2.9) $$\pounds\pounds h<0,\quad h=W_{kjih} W^{kjih}$$

then M is projectively flat if and only if M is isometric to a sphere.

PROOF. We have

$$W_{kjih} W^{kjih}=Z_{kjih} Z^{kjih}-\frac{2}{n-1} G_{ki} G^{ki}$$

Therefore,

$$\pounds(W_{kjih} W^{kjih})=-4\rho W_{kjih} W^{kjih}-\frac{4n}{n-1}(\nabla^j \rho^i)G_{ji}$$

Multiplying both sides by ρ and integrating the resulting equation over M, we find

$$4\int_M \rho^2 h\, dv+\int_M \rho\pounds h\, dv=-\frac{4n}{n-1}\int_M \rho(\nabla^j \rho^i)\, G_{ji}\, dv$$

In the light of (2.3) we get

$$4\int_M \rho^2 h\, dv+\int_M \rho\pounds h\, dv=\frac{4n}{n-1}\int_M G_{ji}\,\rho^j \rho^i\, dv$$

Now let us use the Lemma (3.5) of H. Hiramatu [1]:
According to it for the Riemannian manifold M which is isometric to a sphere, we have

$$4\int_M \rho^2 h\, dv+\int_M \rho\,\pounds\, h\, dv=0$$

Using theorem C we have

$$4\int_M \rho^2 h\, dv-\frac{1}{n}\int_M \pounds\pounds\, h\, dv=0$$

In the light of (2.9), We have

$$\int_M \rho^2 h\, dv\leqq 0,$$

from which $\rho^2 h=0$, or $h=0\Rightarrow M$ is projectively flat.

Conversely, let M be projectively flat then $h=0$ and $\int_M G_{ji}\,\rho^j \rho^i\, dv=0$ from

Lemma (3.5) of Haramatu [1] M is isomtric to a sphere.

THEOREM 2.5. *Under the same assumption as in Theorem* (2.4), *if* $K=const.$ *and* (2.9) *is replaced by*

$$(2.10)\qquad \pounds\left\{\Sigma_{n=o}^{l}\alpha_n\left(-\frac{n-1}{K}\right)^n\Delta^n(\pounds h)\right\}\leqq 0,$$

l being a non-negative integer and α_k *constants such that* $\Sigma_{k=o}^{l}\alpha_k>0$, *then M is projectively flat if and only if M is isometric to a sphere.*

PROOF. Let M be isometric to a sphere then as in the previous theorem we did, we have

$$\begin{aligned}0=4\int_M(\alpha_0+\alpha_1+\cdots+\alpha_l)\,\rho^2\,h\,dv\\ +\int_M\rho\Big\{\alpha_0\pounds h+\alpha_1\Big(-\frac{n-1}{k}\Big)\Delta(\pounds h)+\cdots\\ \cdots+\alpha_l\Big(-\frac{n-1}{k}\Big)^l\Delta^l(\pounds h)\Big\}\,dv\end{aligned}$$

or

$$\begin{aligned}4\int_M(\alpha_0+\alpha_1+\cdots\alpha_l)\,\rho^2\,hdv-\frac{1}{n}\int_M\pounds\Big\{\alpha_0\,\pounds h+\alpha_1\Big(-\frac{n-1}{k}\Big)\Delta(\pounds h)+\cdots\\ \cdots+\alpha_l\Big(-\frac{n-1}{k}\Big)^l\Delta^l(\pounds h)\Big\}dv=0\end{aligned}$$

α_0, α_1, $\cdots$, α_l being constants such that $\Sigma_{k=0}^{l}\alpha_k>0$. Thus by virtue of (2.10) we have $\int_M\rho^2\,h\,dv=0\Rightarrow h=0\Rightarrow W_{kjih}=0$.

Conversely, let M be projectively flat$\Rightarrow h=0\Rightarrow\int_M G_{ji}\,\rho^j\rho^i\,dv\Rightarrow$(from Lemma (3.5) of Hiramatu [1] M is isometric to a sphere.

THEOREM 2.6. *If a compact orientable Riemannian manifold M with constant scalar curvature K and of dimension* $n>2$ *admits an infinitesimal non-isometric conformal transformation* v^h: $\pounds g_{ji}=2\rho\,g_{ji}$, $\rho\neq 0$, *then*

$$\begin{aligned}\int_M\pounds\pounds\Delta(\alpha Z_{kjih}\,Z^{kjih}+\beta W'_{kjih}\,W'^{kjih})\,dv\\ \leqq 4n\int_M\rho^2\,\Delta(\alpha\,Z_{kjih}\,Z^{kjih}+\beta\,W'_{kjih}W'^{kjih})\,dv\end{aligned}$$

for any non-negative constants α *and* β *not both zero, equality holding if and only if M is isometric to a sphere.*

Before proving the above theorem, let us use the following lemma:

LEMMA 2.2. (Yano and Sawaki, [3]). *If a compact orientable Riemannian man-*

ifold M with constant scalar curvature K and of dimesion n admits an infinitesimal conformal transformation v^h: $\mathcal{L} g_{ji}=2\rho g_{ji}$, *then*

$$\int_M \mathcal{L}\mathcal{L}(W'_{kjih} W'^{kjih})dv=-8n(a+b)^2\int_M G_{ji}\rho^j\rho^i\,dv$$
$$+4n\int_M \rho^2 W'_{kjih} W'^{kjih}\,dv.$$

PROOF of Theorem.

$$\int_M \mathcal{L}\mathcal{L}\Delta(\alpha Z_{kjih} Z^{kjih}+\beta W'_{kjih} W'^{kjih})\,dv$$
$$-4n\int_M \rho^2\Delta(\alpha Z_{kjih} Z^{kjih}+\beta W'_{kjih} W'^{kjih})\,dv$$
$$=\{\alpha+\beta(a+b)^2\}\frac{8nK}{n-1}\int_M G_{ji}\rho^j\rho^i\,dv$$

The result now follows from Lemma (3.5) of Hiramatu [1].

REMARK. In 1967 Hsuing introduced a tensor $(a\, Z_{kjih}+b\, g_{kh} G_{ji})$ and obtained a condition for M to be isometric to a sphere (see [2]). After one year in 1968, Yano and Sawaki used a new tensor given below

$$W'_{kjih}=a Z_{kjih}+\frac{b}{n-2}(g_{kh} G_{ji}-g_{jh} G_{ki}+G_{kh} g_{ji}-G_{jh} g_{ki})$$

(vide see Yano and Sawaki [3]). a and b used above are constants. Using W'_{kjih} Yano and Sawaki proved a theorem similar to Hsuing. If we introduce a third tensor:

$$T_{kjih}=a Z_{kjih}+\frac{b}{n-1}[g_{kj} G_{hi}-g_{hj} G_{ki}],$$

then T_{kjih} will possess the properties similar to W'_{kjih} defined by Yano and Sawaki. In short we can easily check that

$$T_{kjih} g^{kh}=(a+b)G_{ji}$$

and that, when $a+b=0$

$$T_{kjih}=a W_{kjih}$$

Where W_{kjih} is projective curvature tensor. We can prove in this case a Lemma similar to lemma (2.1). Thus under the conditions given in Lemma (2.1) we have

$$\int_M G_{ji}\rho^j\rho^i dv=\frac{1}{\alpha'}\int_M\left[\rho^2 T_{hkji} T^{hkji}+\frac{1}{4}\rho\,\mathcal{L} T_{hkji} T^{hkji}\right]dv \tag{2.11}$$

where

$$\alpha'=\left(2a^2+\frac{b(b+2a)(n-2)}{(n-1)}\right)$$

$$=\frac{(n-2)(a+b)^2+na^2}{n-1}$$

Here we can see that for $a=1$, $b=-1$, (2.11) is same as (2.1).

Since for $n>2$, $\alpha'>0$, we can find under the conditions of Theorem (2.1) that if

$$\mathcal{L}\mathcal{L}T_{kjih}\ T^{kjih}\leqq 0$$

Then M is isometric to a sphere. We can check that Theorem (2.1) is a particular case on taking $a=1$, $b=-1$.

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