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PRIMARY IDEALS IN THE RING OF COTINUOUS FUNCTIONS

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0. Abstract

Considering the prime z-filters on a topological space X through the structures of the ring $C(X)$ of continuous functions, a prime z-filter is uniquely determined by a primary z-ideal in the ring $C(X)$, i.e., they have a one-to-one correspondence. Any primary ideal is contained in a unique maximal ideal in $C(X)$. Denoting $\mathscr{P}(X)$, $\mathscr{Q}(X)$, $\mathfrak{M}(X)$ the prime, primary-z, maximal spectra, respectively, $\mathscr{Q}(X)$ is neither an open nor a closed subspace of $\mathscr{P}(X)$.

1. Preliminaries

If X is a topological space, the set $C(X)$ of real continuous functions is a ring. For any z-ideal I in $C(X)$, the following are equivalent: (1) I is prime, (2) I contains a prime ideal, (3) For all $f,g\in C(X)$, if $fg=0$, then $f\in I$ or $g\in I$ (4) For every $f\in C(X)$, there is a zero-set in $Z[I]$ on which f does not change sign, Let $\mathscr{P}(X)$ ($\mathfrak{M}(X)$) be the set of all prime (maximal) ideals in $C(X)$, let $U(E)$ be the set of all prime (maximal) ideals containing the set E, then (i) if a is the ideal generated by E, $U(E)=U(a)=U(r(a))$ (ii) $U(0)=\mathscr{P}(X)$ ($\mathfrak{M}(X)$), $U(1)=\phi$ (iii) if $(E_i)_{i\in I}$ is any family of subsets of $C(X)$, then $U(\bigcup_{i\in I} E_i)=\bigcap_{i\in I} U(E_i)$, (iv) $U(a\cap b)=U(a)\cup U(b)=U(ab)$ for any ideals a,b of $C(X)$. Hence the sets $U(E)$ satisfy the axoms for closed sets in a topological space. The resulting space is said to be the prime (maximal or structure space) spectrum of $C(X)$, respectively.

3. Results

THEOREM 1. *For any z-ideal I in $C(X)$, the following are equivalent:*

(1) *I is primary*

(2) *I contains a primary ideal*

(3) *For all $f,g\in C(X)$, if $fg=0$, then either $f\in I$ or $g^n\in I$ for some $n>0$.*

(4) *For every $f\in C(X)$, there is a zero-set in $Z[I]$ on which f does not change sign.*

LEMMA 2. *If J and J' are primary ideals, neither containing the others, then $J \cap J'$ is not primary*

THEOREM 3. *In $C(X)$, every primary ideal is contained in a unique maximal ideal.*

PROOF. We know that each ideal is contained in at least one maximal ideal. If M and M' are distinct maximal ideals, then $M \cap M'$ is not primary Since M and M' are z-ideals, $M \cap M'$ is also a z-ideal. By Theorem 1, $M \cap M'$ contains no primary ideal.

This is a speciality of characterizations of ideals in $C(X)$, the next theorem shows that a primary ideal uniquely determines a prime z-filter on a topological space X. Therefore we see the correspondence between primary z-ideals and prime z-filters be one-to-one.

THEOREM 4. *If a is primary ideal in $C(X)$, $Z[a]$ is a prime z-filter*

PROOF. Let $a'=Z^{\leftarrow}Z[a]$, then $Z[a']=Z[a]$ and a' is a z-ideal which contains the primary ideal a. By theorem 1, a' is primary z-ideal and hence a' is prime. Thus $Z[a']=Z[a]$ is a prime z-filter.

LEMMA 5. *For all ideal I, we have $Z[I]=Z[r(I)]$*

THEOREM 6. *Let $\mathscr{Q}(X)$ be the set of all primary z-ideals in $C(X)$ and let $V(E)$ be the set of all primary z-ideals which contain E, where E is a subset of $C(X)$. Then* (i) *if a is the ideal generated by E, then $V(a)=V(r(a))=V(E)$,* (ii) $V(0)=\mathscr{Q}(X)$, $V(1)=\phi$ (iii) *if $(E_i)_{i\in I}$ is any family of subsets of $C(X)$, then $V(\bigcup_{i\in I} E_i)=\bigcap_{i\in I} V(E_i)$* (iv) $V(a\cap b)=V(ab)=V(a)\cup V(b)$

Theorem 6 shows that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. We shall say this topological space the primary z-spectrum of $C(X)$.

LEMMA 7. *Let X_f denote the complement of $V((f))$, (f) the ideal generated by f, then the sets X_f form a basis of open sets for the topology and* (i) $X_f\cap X_g =X_{fg}$ (ii) $X_f=\phi$ *if f is nilpotent* (iii) $X_f=\mathscr{Q}(X)$ *if, and only if f is unit* (iv) *$\mathscr{Q}(X)$ is quasi-compact* (v) *More generally X_f is quasi-compact* (vi) *An open subset of $\mathscr{Q}(X)$ is quasi-compact if, and only if it is a finite union of sets X_f*

Furthermore, $\mathscr{Q}(X)$ is a subspace of prime spectrum $\mathscr{P}(X)$ of $C(X)$ and has a maximal spectrum $\mathfrak{M}(X)$ as a subspace. And the sets $V_f=\{a\in\mathscr{Q}(X)|f\in a\}$ form the basis for the closed sets. For a subset $\mathfrak{A}$ of $\mathscr{Q}(X)$ $Cl_{Q(X)}\ \mathfrak{A}=\{a\in\mathscr{Q}(X)|\ a\supset\bigcap\mathfrak{A}\}$ this immediately implies that $\mathscr{Q}(X)$ is not T_1 in general and $\mathscr{Q}(X)$ is neither an open nor a closed subspace of $\mathscr{P}(X)$. But if X is discrete then $\mathscr{Q}(X)=\mathscr{P}(X)$. For the fact that every ideal is a z-ideal, since letting $Z(f)=Z(g)$ for some $g\in I$

Define $h(x)=\begin{cases}\dfrac{f(x)}{g(x)} & if\ x\notin Z(g)\\ 0 & \text{otherwise}\end{cases}$ then $h\in C(X)$ and $f=gh$ and hence $f\in I$

What ever space X does $\mathscr{Q}(X)$ a closed (or an open) subspace of $\mathscr{P}(X)$ make?

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