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A NOTE ON PERIPHERALLY $\mathfrak{M}$-PARACOMPACT SPACES

By M.K. Singal and Shashi Prabha Arya

In [1] E.E. Grace introduced the concept of peripherally paracompact spaces. In the present paper we introduce and study peripherally $\mathfrak{M}$-paracompact spaces. Also, by making use of some other concepts introduced by E.E. Grace [1], we obtain some characterisations of $\mathfrak{M}$-paracompact spaces. A result due to D.R. Traylor [4] for paracompactness in regular spaces, has also been extended to $\mathfrak{M}$-paracompactness in normal spaces.

DEFINITION 1. A family $\mathscr{A}$ of open subsets of a space X is said to have property $\mathscr{P}$ in the strong sense (resp. in the weak sense) if $\mathscr{A}$ has the property $\mathscr{P}$ as a collection of open sets in X (resp. in the subspace $\bigcup\{A : A\in\mathscr{A}\}$ of X).

DEFINITION 2. A space X is said to be *peripherally $\mathfrak{M}$-paracompact* in the strong sense (resp. in the weak sense) if for each frontier set (that is, each nowhere dense, closed set) F in X and each open covering $\mathscr{U}$ of X of cardinality $\leq\mathfrak{M}$, there is an open refinement $\mathscr{V}$ of $\mathscr{U}$, covering F, which is locally finite in the strong sense (resp. in the weak sense).

THEOREM 1. *A space X is $\mathfrak{M}$-paracompact if and only if it is peripherally $\mathfrak{M}$-paracompact in the strong sense.*

PROOF. Only the if part need be proved. Let $\mathscr{G}$ be any open covering of X of cardinality $\leq\mathfrak{M}$. Let $\mathscr{H}$ be a family of mutually disjoint open sets refining $\mathscr{G}$ such that $H^*=\bigcup\{H: H\in\mathscr{H}\}$ is dense in X. Then, $X\sim H^*$ is a nowhere, closed set. Let $\mathscr{E}$ be a locally finite, open refinement of $\mathscr{G}$ covering the frontier set $X\sim H^*$ and let $\mathscr{A}$ be a locally finite, open refinement of $\mathscr{G}$ covering the boundary of $E^*=\bigcup\{E: E\in\mathscr{E}\}$. Consider now, the family $\mathscr{H}'=\{H\cap(X\sim\overline{E}^*): H\in\mathscr{H}\}$. It is easy to verify that $\mathscr{H}'$ is a discrete family of open sets and that $\mathscr{H}'\cup\mathscr{E}\cup\mathscr{A}$ is a locally finite open refinement of $\mathscr{G}$ which covers X and hence X is $\mathfrak{M}$-paracompact.

THEOREM 2. *A normal space X is peripherally $\mathfrak{M}$-paracompact in the strong sense iff it is peripherally $\mathfrak{M}$-paracompact in the weak sense.*

PROOF. Let $\mathscr{C}$ be any open covering of X of cardinality $\leq\mathfrak{M}$ and let F be any frontier subset of X. If X is peripherally $\mathfrak{M}$-paracompact in the weak sense, then there exists an open refinement $\mathscr{H}$ of $\mathscr{C}$ covering F which is locally finite at each point of $H^*=\bigcup\{H: H\in\mathscr{H}\}$. Since X is normal, and F and $X\sim H^*$ are disjoint closed sets, therefore exists an open set W: $F\subset W\subset\overline{W}\subset X\sim H^*$. Let $\mathscr{W}=\{H\cap W: H\in\mathscr{H}\}$. Then $\mathscr{W}$ is a locally finite open refinement of $\mathscr{C}$ which covers F and hence X is peripherally $\mathfrak{M}$-paracompact in the strong sense.

DEFINITION 3. A family $\mathscr{F}$ of continuous functions on a space X into the non-negative real numbers is called a *partition of unity* on X if for each point $x\in X$, $\sum f(x)=1$. $\mathscr{F}$ is said to be *subordinated* to a covering $\mathscr{U}$ of X if for each $f\in\mathscr{F}$, $f(X\sim U)=\{0\}$ for some $U\in\mathscr{U}$.

THEOREM 3. *A normal space X is $\mathfrak{M}$-paracompact iff for every open covering $\mathscr{C}$ of X of cardinality $\leq\mathfrak{M}$ and for every frontier set F, there exists an open refinement $\mathscr{H}$ of $\mathscr{C}$, covering F and which has a partition of unity subordinated to it in the weak sense.*

RROOF. To prove the 'if' part, let $\mathscr{C}$ be any open covering of X of cardinality $\leq\mathfrak{M}$. Let $\mathscr{H}$ be a family of disjoint open sets refining $\mathscr{C}$ such that $H^*=\bigcup\{H: H\in\mathscr{H}\}$ is dense in X. Then $X\sim H^*$ is a frontier set. By hypothesis, there exists an open refinement $\mathscr{W}$ of $\mathscr{C}$ which covers $X\sim H^*$ and which has a partition of unity Φ subordinated to it in the weak sense. Since X is normal, and $X\sim H^*$ and $X\sim\bigcup\{W: W\in\mathscr{W}\}$ are disjoint closed sets, therefore, there exists a continuous function g: $X\to[0, 1]$ such that $g(X\sim H^*)=\{1\}$ and $g(X\sim\bigcup_{W\in\mathscr{W}}W)=\{0\}$. For each $f\in\Phi$, let $f'(x)=f(x)\cdot g(x)$ for $x\in\bigcup\{W: W\in\mathscr{W}\}$ and let $f'(x)=0$ for $x\in X\sim\bigcup\{W: W\in\mathscr{W}\}$. For each $H\in\mathscr{H}$, there exists a continuous function: g_H: $X\to[0, 1]$ such that $g_H(X\sim H)=\{0\}$ and $g_H(H-g^{-1}(0))=\{1\}$. Let h be defined as

$$h(x)=\begin{cases}\sum_{f\in\Phi}f'(x), & \text{if } x\in X\sim H^*\\ \sum_{f\in\Phi}f'(x)+g_H(x), & \text{if } x\in H^*.\end{cases}$$

Then $\mathscr{C}$ has the partition of unity $\Phi=\{f'/h: f\in\Phi\}\cup\{g_H/h: H\in\mathscr{H}\}$ subordinated to it. Thus, every open covering of X of cardinality $\leq\mathfrak{M}$ has a partition of unity subordinated to it and hence X is $\mathfrak{M}$-paracompact [2, theorem 2]. Converse is obviously true, [2, theorem 2].

THEOREM 4. *For a normal space X, the following are equivalent*:

(*a*) *X is $\mathfrak{M}$-paracompact.*

(*b*) *For every covering $\mathscr{U}$ of X of cardinality $\leq \mathfrak{M}$ and for each frontier set F in X, there is an open refinement $\mathscr{V}$ of $\mathscr{U}$, covering F, such that $\mathscr{V}$ is cushioned in $\mathscr{U}$ in the strong sense.*

(*c*) *For every open covering $\mathscr{U}$ of X of cardinality $\leq \mathfrak{M}$ and for each frontier set F in X, there is an open refinement $\mathscr{V}$ of $\mathscr{U}$ covering F, such that $\mathscr{V}$ is cushioned in $\mathscr{U}$ in the weak sense.*

(*d*) *For every open covering $\mathscr{U}$ of X of cardinality $\leq \mathfrak{M}$ and for each frontier set F in X, there is an open refinment $\mathscr{V}$ of $\mathscr{U}$ covering F, such that $\mathscr{V}$ is σ-cushioned in $\mathscr{U}$ in the weak sense.*

(*e*) *For each every open covering $\mathscr{U}$ of X of cardinality $\leq \mathfrak{M}$ and for each frontier set F in X, there is an open refinement $\mathscr{V}$ of $\mathscr{U}$ covering F, such that $\mathscr{V}$ is σ-cushioned in $\mathscr{U}$ in the strong sense.*

PROOF. $(a) \Longrightarrow (b)$. Every open covering $\mathscr{U}$ of X of cardinality $\leq \mathfrak{M}$ will have an open, cushioned refinement in view of Theorem 1 and hence (b) is true.

$(b) \Longrightarrow (c)$ Obvious

$(c) \Longrightarrow (d)$ Obvious

$(d) \Longrightarrow (e)$. Since X is normal, a proof similar to theorem 2 applies.

$(e) \Longrightarrow (a)$. This follows in a manner similar to the proof of theorem 1

DEFINITION 4. A space X is said to be *$\mathfrak{M}$-paracompact in a discrete peripheral sense* if for every open covering $\mathscr{U}$ of X of cardinality $\leq \mathfrak{M}$ there exists an open refinement $\mathscr{V}$ of $\mathscr{U}$ such that if $\mathscr{F}$ be any discrete family of closed set refining $\mathscr{V}$, then the boundary of $\cup\{F\colon F \in \mathscr{F}\}$ is $\mathfrak{M}$-paracompact with respect to the space X.

DEFINITION 5. A space X is said to be *subparacompact* if for every open covering $\mathscr{G}$ of X, there exists a sequence $\{\mathscr{F}_i\colon i=1, \cdots\}$ of discrete families of closed sets such that $\bigcup_{i=1}^{\infty} \mathscr{F}_i$ is a refinement of $\mathscr{G}$.

THEOREM 4. *If X is a normal, subparacompact space which is countably paracompact in a discrete peripheral sense, then X is countably paracompact.*

PROOF. Essentially the same as that of ([4], theorem 5) Traylor states the theorem with 'semi-method 'instead of 'subparacompact'. However, while proving the theorem, only subparacompactness is being used. It should be noted that every normal, semi-metric space is perfectlynormal and a perfectly normal space is always countably paracompact. So the theorem becomes obvious with subparacompact replaced by semi-metric.

THEOREM 5. *If X is a normal, subparacompact space which is $\mathfrak{M}$-paracompact in a discrete peripheral sense, then X is $\mathfrak{M}$-paracompact.*

PROOF. Since X is $\mathfrak{M}$-paracompact in a discrete peripheral sense, therefore, X is countably paracompact in a discrete peripheral sense. Then X is countably paracompact by theorem 4. Now, let $\mathscr{U}=\{U_\alpha: \alpha\in\Lambda\}$ be any open covering of X of cardinality $\leq\mathfrak{M}$ Let Λ be well ordered by $<$. Let $\mathscr{U}'$ be an open refinement of $\mathscr{U}$ covering X such that the boundary of the union of each discrete family of closd sets refining $\mathscr{U}$ is $\mathfrak{M}$-paracompact with respect to X. Since X is subparacompact, there exists a sequence $\{\mathscr{F}_i: i\in N\}$ of discrete families of closed sets. For each $\alpha\in\Lambda$, let $\mathscr{F}_{1\alpha}$ denote the subfamily of $\mathscr{F}_1$ consisting of all sets $G\in\mathscr{F}_1$ for which α is the first index such that $G\subset U_\alpha$. If $G\in\mathscr{F}_{1\alpha}$ for some α, denote by V_G an open set which contains boundary of G such that $V_G\supset U_\alpha$ and V_G does not intersect $[(\cup\{F: F\in\mathscr{F}_1\})\sim G]$. Denote by $\mathscr{V}_{1\alpha}$ the family consisting of all sets V such that there exists $G\in\mathscr{F}_{1\alpha}$ such that $V=V_G$. Since boundary of $\cup\{F: F\in\mathscr{F}_1\}$ is $\mathfrak{M}$-paracompact and $\mathscr{V}_1=\bigcup_{\alpha\in\Lambda}\mathscr{V}_{1\alpha}$ is a covering of the boundary of $\cup\{F: F\in\mathscr{F}_1\}$; therefore, there exists a locally finite open refinement $\mathscr{V}_1'$ of $\mathscr{V}_1$ such that $\mathscr{V}_1'$ covers boundary of $\cup\{F: F\in\mathscr{F}_1\}$. Now, denote by $\mathscr{V}_1''$ the family consisting of all sets V for which there is a $G\in\mathscr{F}$ such that $x\in V$ iff either $x\in G$ or x is a point of a member of $\mathscr{V}_1'$ which intersects G. Clearly, $\mathscr{V}_1''$ is an open refinement of $\mathscr{U}'$ which covers $\cup\{F: F\in\mathscr{F}_1\}$. Now consider $\mathscr{F}_2$. Denote by $\mathscr{F}_2'$ the family consisting of all sets G such that there exists $H\in\mathscr{F}_2$ such that $G=H\sim[H\cap(\cup\{V: V\in\mathscr{V}_1''\})]$. Clearly, $\mathscr{F}_2'$ is discrete family of closed sets refining $\mathscr{U}'$. For each $\alpha\in\Lambda$, denote by $\mathscr{F}_{2\alpha}$ the subfamily of $\mathscr{F}_2'$ consisting of only those sets each of which is a subset of U_α but none is a subset of U_β for $\beta<\alpha$. If $G\in\mathscr{F}_{2\alpha}$, denote by V_G an open set containing the boundary of G such that $H_\alpha\subset V_G$, V_G does not intersect $[\cup\{F: F\in\mathscr{F}_2\}\sim G]$. Let $\mathscr{V}_{2\alpha}$ denote the family consisting of all sets V for which there is a $G\in\mathscr{F}_{2\alpha}$ such that $V=V_G$. Let $\mathscr{V}_2=\bigcup_{\alpha\in\Lambda}\mathscr{V}_{2\alpha}$. As before, there exists a locally finite, open refinement $\mathscr{V}_2'$

of $\mathscr{V}_2$ which covers the boundary of $\bigcup\{F: F\in\mathscr{F}_2\}$ and thus there is a locally finite open refinement $\mathscr{V}_2''$ of $\mathscr{U}'$ such that $\mathscr{V}_2''$ covers $\bigcup\{F: F\in\mathscr{F}_2'\}$. This process may be continued indefinitely as follows: for each positive integer $n>2$, denote by $\mathscr{F}_n'$ the collection which consists of all sets G for which there is a $H\in\mathscr{F}_n$ such that $G=H\sim(H\cap[\bigcup\{V: V\in\mathscr{V}_i'', i=1,\cdots n-1\}])$. Clearly, $\mathscr{F}_n'$ is a discrete family of closed sets such that $\mathscr{F}_n'$ refines $\mathscr{U}'$. As before, denote by $\mathscr{F}_{2\alpha}$ the subfamily of $\mathscr{F}_n'$ consisting of just those sets each of which is a subset of U_α, but none is a subset of U_β for $\beta<\alpha$. For $G\in\mathscr{F}_n^\alpha$, let V_G denote an open set containing the boundary of G such that $V_G\supset U_\alpha$, V_α does not intersect $(\bigcup\{F: F\in\mathscr{F}_i, i=1,\cdots, n-1\}$ and also does not intersect $(\bigcup\{F: F\in\mathscr{F}_i\})\sim G$. If $\mathscr{V}_{nk}$ denotes the family consisting of those sets $G\in\mathscr{F}_{n\alpha}$ such that $V=V_G$ and if $\mathscr{V}_n=\bigcup_{\alpha\in A}\mathscr{V}_{n\alpha}$, then there exists a locally finite, open refinement $\mathscr{V}_n'$ of $\mathscr{V}_n$ such that $\mathscr{V}_n'$ covers the boundary of $\bigcup\{F: F\in\mathscr{F}_n'\}$ and thus there is a locally finite open refinement $\mathscr{V}_n''$ of $\mathscr{U}'$ such that $\mathscr{V}_n''$ covers $\bigcup\{F: F\in\mathscr{F}_n'\}$. Now, $\bigcup_{n=1}^{\infty}\mathscr{V}_n''$ is a σ-locally finite, open refinement of $\mathscr{U}'$ and hence of $\mathscr{U}$. Thus every open covering of X of cardinality $\leq\mathfrak{M}$ has a σ-locally finite open refinement. Also, X is a countably paracompact. Therefore X is $\mathfrak{M}$-paracompact ([3], theorem 5).

COROLLARY *Every normal space which is either semi-metric or developable or Moore, and is $\mathfrak{M}$-paracompact in a discrete peripheral sense, is $\mathfrak{M}$-paracompact.*

PROOF. Every semi-metric, or developable, or Moore space is subparacompact and hence the result follows from theorem 5.

Institute of Advanced Studies,
Meerut University, Meerut
and
Maitreyi College, University of Delhi
India

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