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SPACES IN WHICH THE CLOSURE OF A COMPACT SET IS COMPACT

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1. Introduction

Well known conditions for a space to have the property that compact subsets have compact closures are: compactness, regularity, Hausdorff (theorem 2.2). Spaces with this property are called C-spaces. Example 7.1 shows that T_1 is not a sufficient condition for a space to be a C-space.

In theorem 2.5, we relax the compactness of X and show that X is a C-space if the derived set of X is compact. We introduce the concept of weakly Hausdorff and show that it is a sufficient condition for a space to have property C (theorem 2.7). Normal and metacompactness together imply property C (theorem 2.8).

Closed subspaces of C-spaces are shown to be C-spaces (theorem 3.1) and disjoint sums of C-spaces are shown to be C-spaces (corollary 3.4). A sufficient condition is given for the intersection of two C-sets to be a C-set (theorem 3.5). Example 7.7 shows that in general, the intersection of two C-sets need not be a C-set. A product space is shown to be a C-space if and only if each factor space is a C-space (theorem 4.1).

In theorem 5.2, a necessary and sufficient condition is given for a simple extension of a topology to be a C-topology.

If $f:(X, \mathscr{T})\to(Y, \mathscr{U})$ is a surjection and $\mathscr{T}$ is the weak topology, then $\mathscr{T}$ is a C-topology if and only if $\mathscr{U}$ is a C-topology (theorem 6.1).

In §7, examples are given relative to infima and suprema of C-topologies and intersections of C-subsets of a space.

2. Sufficient conditions

DEFINITION 2.1. A space $(X,\mathscr{T})$ will be called a *C-space* and $\mathscr{T}$ will be called a C-topology iff for each compact set $K\subset X$, then $c(K)$ is compact, c denoting the closure operator. A subset $A\subset X$ is called a C-set iff $(A, A\cap\mathscr{T})$ is a C-space.

We list the well known results of such spaces in

THEOREM 2.2. $(X, \mathscr{T})$ *is a C-space if any one of the following hold*:

(i) $(X, \mathscr{T})$ *is compact* (ii) $(X, \mathscr{T})$ *is regular* (iii) $(X, \mathscr{T})$ *is Hausdorff.*

We shall weaken conditions (i) and (iii) to get theorems 2.5 and 2.7. But first we prove two lemmas.

LEMMA 2.3. *A space* $(X, \mathscr{T})$ *is a C-space iff* $F\subset c(K)-K$ *implies that F is compact when F is closed and K is compact.*

PROOF. Let $(X, \mathscr{T})$ be a C-space and $F\subset c(K)-K$. Then $F\subset c(K)$ and $c(K)$ is compact. Thus F is compact, being a closed subset of $c(K)$.

Conversely, let $K\subset X$, K compact. Suppose $c(K)\subset\cup\{O_\alpha : \alpha\in\Delta\}$, $O_\alpha\in\mathscr{T}$. Since K is compact, there exist α_i such that $K\subset O_{\alpha_1}\cup\cdots\cup O_{\alpha_n}$. Let $F=c(K)-(O_{\alpha_1}\cup\cdots\cup O_{\alpha_n})$. Then $F\subset c(K)-K$ and hence F is compact. There exists then $\beta_1, \cdots, \beta_m$ in Δ such that $F\subset O_{\beta_1}\cup\cdots\cup O_{\beta_m}$. It follows then that $c(K)\subset O_{\alpha_1}\cup\cdots\cup O_{\alpha_n}\cup O_{\beta_1}\cup\cdots\cup O_{\beta_m}$.

LEMMA 2.4. *If* $(X, \mathscr{T})$ *is compact, then* X' *is compact,* X' *denoting the derived set of* X.

PROOF. Let $x\notin X'$; then $\{x\}\in\mathscr{T}$ and X' is closed.

THEOREM 2.5. *Let* $(X, \mathscr{T})$ *be a space and suppose that* X' *is compact. Then* $(X, \mathscr{T})$ *is a C-space.*

PROOF. We employ lemma 2.3: let $F\subset c(K)-K$, F being closed and K being compact. Then $F\subset K'\subset X'$ and $F\subset X'$. It follows then that F is compact.

To obtain a generalization of (iii) in theorem 2.2, we introduce

DEFINITION 2.6. We say that a space $(X, \mathscr{T})$ is *weakly Hausdorff* iff $c(x)=c(y)$ whenever there exists a net $S: D\to X$ for which $\lim S=x$ and $\lim S=y$.

THEOREM 2.7. *If* $(X, \mathscr{T})$ *is weakly Hausdorff, then* $(X, \mathscr{T})$ *is a C-spaces.*

PROOF. Let $c(K)\subset\cup\{O_\alpha : \alpha\in\Delta\}$, K compact and $O_\alpha\in\mathscr{T}$. Then $K\subset O_{\alpha_1}\cup\cdots\cup O_{\alpha_n}$ for some $\alpha_i\in\Delta$. Let $x\in c(K)$. There exists then a net $S: D\to K$ such that $\lim S=x$. Since K is compact, there exists a subset $T: E\to K$ and a point $y\in K$ such that $\lim T=y$. Since $\lim T=x$, it follows that $c(x)=c(y)$. Now $y\in O_{\alpha_i}$ for some i and hence $x\in O_{\alpha_i}$. Thus $c(K)\subset O_{\alpha_1}\cup\cdots\cup O_{\alpha_n}$ and $c(K)$ is compact.

THEOREM 2.8. *Let* $(X, \mathscr{T})$ *be normal and metacompact. Then* $(X, \mathscr{T})$ *is a C-space.*

PROOF. Let $K \subset X$, K compact and suppose that $c(K) \subset \bigcup \{O_\alpha : \alpha \in \Delta\}$. Then $X = \bigcup \{O_\alpha : \alpha \in \Delta\} \cup \{\mathscr{C}c(K)\}$. Since $(X, \mathscr{T})$ is metacompact, there exists an open point-finite refinement $\{O_\gamma : \gamma \in \Gamma\}$ of $\{O_\alpha : \alpha \in \Delta\} \cup \{\mathscr{C}c(K)\}$. But X is normal and hence there exists an open cover $\{O^*_\gamma : \gamma \in \Gamma\}$ of X such that $c(O^*_\gamma) \subset O_\gamma$ for each $\gamma \in \Gamma$. Now $K \subset O^*_{\gamma_1} \cup \cdots \cup O^*_{\gamma_n}$ and hence $c(K) \subset O_{\gamma_1} \cup \cdots \cup O_{\gamma_n}$. We may assume that $O_{\gamma_i} \not\subset \mathscr{C}c(K)$ for each i. Hence $O_{\gamma_i} \subset O_{\alpha_i}$ for some α_i and $c(K)$ is compact.

COROLLARY 2.9. *If $(X, \mathscr{T})$ is normal and paracompact, then $(X, \mathscr{T})$ is a C-space.*

3. Subspaces

THEOREM 3.1. *Let $(Y, \mathscr{U})$ be a closed subspace of a C-space $(X, \mathscr{T})$. Then $(Y, \mathscr{U})$ is a C-space.*

PROOF. Let $K \subset Y$, K compact; then $c(K)$ is compact and hence $Y \cap c(K)$ is compact, Y being closed. Thus $c_Y(K)$ is compact and $(Y, \mathscr{U})$ is a C-space.

THEOREM 3.2. *Let $(X, \mathscr{T})$ be a space and $\{F_\alpha : \alpha \in \Delta\}$ a locally finite closed cover of X. Then $(X, \mathscr{T})$ is a C-space iff $(F_\alpha, F_\alpha \cap \mathscr{T})$ is a C-space for each $\alpha \in \Delta$.*

PROOF. The necessity follows from theorem 3.1. To show the sufficiency, let $K \subset X$, K compact. Since $\{F_\alpha : \alpha \in \Delta\}$ is locally finite, and K is compact, there exists an $O \in \mathscr{T}$ such that $K \subset O$ and $O \cap F_{\alpha_i} \neq \phi$ for $\alpha_1, \cdots, \alpha_n$ only. It follows then that $K = \bigcup \{K \cap F_{\alpha_i} : 1 \leqq i \leqq n\}$ and $c(K) = \bigcup \{c(K \cap F_{\alpha_i}) : 1 \leqq i \leqq n\} = \bigcup \{F_{\alpha_i} \cap c(K \cap F_{\alpha_i}) : 1 \leqq i \leqq n\} = \bigcup \{c_{\alpha_i}(K \cap F_{\alpha_i}) : 1 \leqq i \leqq n\}$. But $K \cap F_{\alpha_i}$ is compact and hence $c_{\alpha_i}(K \cap F_{\alpha_i})$ is compact since F_{α_i} is a C-space. It follows then that $c(K)$ is compact, being a finite union of compact sets.

COROLLARY 3.3. *Let $(X, \mathscr{T})$ be a space and $X = \bigcup \{O_\alpha : \alpha \in \Delta\}$ where $O_\alpha \in \mathscr{T}$ and $O_\alpha \cap O_\beta = \phi$ when $\alpha \neq \beta$. Then $(X, \mathscr{T})$ is a C-space iff $(O_\alpha, O_\alpha \cap \mathscr{T})$ is a C-space for each $\alpha \in \Delta$.*

PROOF. $\{O_\alpha : \alpha \in \Delta\}$ is a locally finite family of closed sets.

COROLLARY 3.4 *Let $(X, \mathscr{T})$ be a disjoint union of spaces $\{(X_\alpha, \mathscr{T}_\alpha) : \alpha \in \Delta\}$. Then $(X, \mathscr{T})$ is a C-space iff $(X_\alpha, \mathscr{T}_\alpha)$ is a C-space for each $\alpha \in \Delta$.*

PROOF. $\{X_\alpha : \alpha \in \Delta\}$ is a disjoint open cover of X.

THEOREM 3.5. *Let* $(X, \mathscr{T})$ *be a space in which the intersection of two compact sets is compact* (*see* [1]). *If* A and B *are C-subsets of* X, *then* $A \cap B$ *is a C-subset.* (*See exampls* 7.7, 7.8.)

PROOF. Let $K \subset A \cap B$, K compact. Then $c_{A \cap B}(K) = A \cap B \cap c(K) = (A \cap c(K)) \cap (B \cap c(K)) = c_A(K) \cap c_B(K)$. Since $c_A(K)$ and $c_B(K)$ are each compact, it follows that $c_{A \cap B}(K)$ is compact.

4. Product spaces

THEOREM 4.1. *Let* $(X, \mathscr{T}) = \times \{(X_\alpha, \mathscr{T}_\alpha) : \alpha \in \Delta\}$. *Then* $(X, \mathscr{T})$ *is a C-space iff* $(X_\alpha, \mathscr{T}_\alpha)$ *is a C-space for each* $\alpha \in \Delta$.

PROOF. Suppose $(X_\alpha, \mathscr{T}_\alpha)$ is a C-space for each $\alpha \in \Delta$ and let $K \subset X$, K compact. Now $K \subset P_\alpha^{-1} c_\alpha(P_\alpha K)$ for each $\alpha \in \Delta$ and hence $K \subset \times \{c_\alpha(P_\alpha K) : \alpha \in \Delta\}$. But $c_\alpha(P_\alpha K)$ is a closed compact set and by the Tychonoff theorem, $\times \{c_\alpha(P_\alpha K) : \alpha \in \Delta\}$ is a closed compact set. It follows then that $c(K)$ is compact.

Conversely, suppose that $(X, \mathscr{T})$ is a C-space and $K_\beta \subset X_\beta$, K_β compact. Take $x_\alpha \in X_\alpha$ arbitrary for each $\alpha \neq \beta$ and let $K = \times \{A_\alpha : \alpha \in \Delta\}$ where $A_\alpha = \{x_\alpha\}$ if $\alpha \neq \beta$ and $A_\beta = K_\beta$. Then $c(K) = \times \{c_\alpha(A_\alpha) : \alpha \in \Delta\}$ and $c(K)$ is compact since K is compact. Again by the Tychonoff theorem, $c_\beta(K_\beta)$ is compact.

5. Simple extension of a topology

DEFINITION 5.1. Let $\mathscr{T}$ be a topology on a set X and let $A \subset X$, $A \notin \mathscr{T}$. Then $\mathscr{T}[A]$ is defined to be $\mathscr{T} \vee \{\phi, A, X\}$ and is called the *simple extension* of $\mathscr{T}$ by A (see [2]).

THEOREM 5.2. *Let* $(X, \mathscr{T})$ *be a space and suppose that* $F \notin \mathscr{T}$, F *closed. Then* $\mathscr{T}[F]$ *is a C-topology iff* $F \cap \mathscr{T}$ *and* $\mathscr{C}F \cap \mathscr{T}$ *are C-topologies.*

PROOF. In [2], it is proved that $F \cap \mathscr{T} = F \cap \mathscr{T}[F]$ and $\mathscr{C}F \cap \mathscr{T} = \mathscr{C}F \cap \mathscr{T}[F]$. Furthermore, F and $\mathscr{C}F$ are each open relative to $\mathscr{T}[F]$ and hence by corollary 3.3, $(X, \mathscr{T}[F])$ is a C-space iff $(F, F \cap \mathscr{T}[F])$ and $(\mathscr{C}F, \mathscr{C}F \cap \mathscr{T}[F])$ are C-spaces. The theorem then follows.

6. Transfer topologies

THEOREM 6.1. *Let* $f : (X, \mathscr{T}) \to (Y, \mathscr{U})$ *be a surjection with* $\mathscr{T}$ *the weak topology. Then* $\mathscr{T}$ *is a C-topology iff* $\mathscr{U}$ *is a C-topology.*

PROOF. Let $\mathscr{U}$ be a C-topology and suppose that $K\subset X$, K compact. Then $f[K]$ is compact and hence $c(f[K])$ is compact in Y. But $c(K)\subset f^{-1}\ c(f[K])$ and $f^{-1}\ c(f[K])$ is compact. It follows then that $c(K)$ is compact.

Conversely, let $K\subset Y$, K compact. Then $f^{-1}K$ is compact and hence $c(f^{-1}K)$ is compact. Then since f is a closed transformation, $fc(f^{-1}K)\supset c(K)$ and hence $c(K)$ is compact.

Property C is not invariant under continuous open surjections (see example 7.2).

7. Examples

EXAMPLE 7.1. T_1 does not imply property C. Let $X=\{1, 2, \cdots, n, \cdots\}$ and $K=\{1, 3, 5, \cdots\}$. Let $\mathscr{T}=\{O : O=\phi$ or $K\cap\mathscr{C}O$ is finite$\}$. It is easy to see that $\mathscr{T}$ is a T_1 topology for X, K is compact, $c(K)=X$ and X is not compact.

EXAMPLE 7.2. A continuous, open image of a C-space need not be a C-space. In particular, quotient spaces of C-spaces need not be C-spaces. Let $(Y, \mathscr{U})$ be an arbitrary space which is not a C-space. There exists a Hausdorff space $(X, \mathscr{T})$ and a continuous open surjection $f: (X, \mathscr{T})\rightarrow(Y, \mathscr{U})$ (see [3], page 92). By (iii) of theorem 2.2, $(X, \mathscr{T})$ is a C-space.

EXAMPLE 7.3. The intersection of two C-topologies need not be a C-topology. Let $X=\{1, 2, 3, \cdots, n, \cdots\}$, $\mathscr{B}_1=\{\{1\}, \{1,2\}, \{3\}, \{3,4\}, \cdots, \{2n+1\}, \{2n+1, 2n+2\},\cdots\}$, $\mathscr{B}_2=\{\{1\}, \{2, 3\}, \{4, 5\}, \cdots, \{2n, 2n+1\}, \cdots\}$, $\mathscr{T}_1$ generated by $\mathscr{B}_1$ as base and $\mathscr{T}_2$ generated by $\mathscr{B}_2$ as base. Then $\mathscr{T}_1$ and $\mathscr{T}_2$ are each C-topologies since compact sets are finite in each topology and the closures of compact sets are finite. But $\mathscr{T}_1\cap\mathscr{T}_2=\{\{1\}, \{1, 2, 3\}, \{1, 2, 3, 4, 5\}, \cdots\}$ which is not a C-topology since $\{1\}$ is compact, but $c\{1\}=X$ which is not compact.

EXAMPLE. 7.4. An intersection of a chain of C-topologies need not be a C-topology. Let $X=\{1, 2, \cdots, n, \cdots\}$, $\mathscr{T}_1$ be generated by $\{\{x\} : x\in X\}$ as base, $\mathscr{T}_2$ be generated by $\{\{1\}, \{1, 2\}, \{3\}, \{4\}, \cdots, \{n\}, \cdots\}$ as base, $\mathscr{T}_3$ be generated by $\{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{4\}, \{5\}, \cdots, \{n\}, \cdots\}$ as base and $\mathscr{T}_n$ be generated by $\{\{1\}, \{1, 2\}, \cdots, \{1, \cdots, n\}, \{n+1\}, \{n+2\}, \cdots\}$ as base. Then $\mathscr{T}_n$ is a C-topology for each $n\in X$, but $\cap\{\mathscr{T}_n : n\in X\}=\{\phi, \{1\}, \{1, 2\}, \{1, 2, 3\}, \cdots, X\}$ and $\cap\{\mathscr{T}_n : n\in X\}$ is not a C-topology.

EXAMPLE 7.5. The supremum of two C-topologies need not be a C-topology.

Let $X=\{1, 2, \cdots, n, \cdots\}$ and $\mathscr{T}_1=\{\phi, \{1\}, X\}$, $\mathscr{T}_2=\{O : O=\phi$ *or* $O=X$ *or* $2\in O$, $1\notin O\}$. Since $\mathscr{T}_1$ and $\mathscr{T}_2$ are each compact, it follows that each is a C-topology. In $\mathscr{T}_1 \vee \mathscr{T}_2$, $\{2\}$ is compact, but its closure is $\{2, 3, 4, \cdots\}$ which is not compact.

EXAMPLE 7.6. The supremum of a chain of compact topologies need not be a C-topology. Let $X=\{1, 2, \cdots, n, \cdots\}$ and $\mathscr{T}_n=\{\phi, X, \{1\}, \{1, 2\}, \cdots, \{1, \cdots, n\}\}$ for each positive integer n. Then each $\mathscr{T}_n$ is a compact topology, but $\{1\}$ is compact in sup $\mathscr{T}_n$, but the closure of $\{1\}$ is not compact in sup $\mathscr{T}_n$.

EXAMPLE 7.7. An intersection of two C-sets need not be a C-set. Let $(X, \mathscr{T})$ be the space in example 7.1, $Y=X\cup\{a, b\}$ and $\mathscr{U}=\mathscr{T}\cup\{Y\}$; let $A=X\cup\{a\}$ and $B=X\cup\{b\}$. Then A and B are compact subsets of $(Y, \mathscr{U})$ and hence C-spaces, but $A\cap B=X$ which is not a C-set.

EXAMPLE 7.8. An intersection of a chain of C-sets need not be a C-set. Let $(X, \mathscr{T})$ be the space in example 7.1 and let $Y=X\cup\{-1, -2, -3, \cdots, -n, \cdots\}$, $\mathscr{U}=\mathscr{T}\cup\{Y\}$. Let $A_n=X\cup\{-n, -(n+1), \cdots\}$ for each n. Then $\{A_n : n=1, 2, \cdots\}$ is a chain of compact sets, but $\cap\{A_n\}=X$ which is not a C-set.

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