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COVERING CHARACTERIZATION OF LOCALLY UNIFORM SPACES.

By R. Vasudevan and C.K. Goel

Abstract

In this paper locally uniform spaces have been characterized through covers. It has been shown that both the approaches are equivalent.

1. Introduction

The systematic study of locally uniform spaces has been initiated by Williams [3]. A locally uniform space $(X, \mathscr{D})$ is one which is obtained from a uniform space by localizing the triangle inequality. In this paper we develop the study of these spaces through the covering approach as has been done for uniform spaces by Tukey [1]. Tukey defined a uniform space (X,μ) as an ordered pair consisting of a non-empty set X and a collection μ of all covers of X which satisfy the following properties:

(i) $\mathscr{U}_1$, $\mathscr{U}_2 \in \mu$ implies there exists a $\mathscr{U}_3 \in \mu$ such that $\mathscr{U}_3{}^* < \mathscr{U}_1$ and $\mathscr{U}_3{}^* < \mathscr{U}_2$.

(ii) if $\mathscr{U} < \mathscr{V}$ and $\mathscr{U} \in \mu$ implies $\mathscr{V} \in \mu$.

When not otherwise specified, the terminology used in this paper is that of Willard [2]. Now we define some notions we have used in the text.

DEFINITION 1.1. Let X be any non-empty set and suppose $\mathscr{U}$ and $\mathscr{V}$ are any covers of X. Then $\mathscr{U}$ is called a *refinement* of $\mathscr{V}$, symbolically $\mathscr{U} < \mathscr{V}$ if each member of $\mathscr{U}$ is contained in some member of $\mathscr{V}$. Next let $A \subset X$ be any subset of X. The *star* of A with respect to $\mathscr{U}$ is the set

$$\mathrm{St}\,(A,\mathscr{U}) = \cup \{U \in \mathscr{U} : U \cap A \neq \phi\}.$$

$\mathscr{U}$ is said to be a *star refinement* of $\mathscr{V}$, written as $\mathscr{U}^* < \mathscr{V}$ if $\{\mathrm{St}\,(U,\mathscr{U}) : U \in \mathscr{U}\} < \mathscr{V}$. $\mathscr{U}$ is called a *barycentric refinement* of $\mathscr{V}$, written as $\mathscr{U}^{\Delta} < \mathscr{V}$, if $\{\mathrm{St}(x,\mathscr{U}) : x \in X\} < \mathscr{V}$, where $\mathrm{St}\,(x,\mathscr{U}) = \cup \{U \in \mathscr{U} : x \in U\}$.

DEFINITION 1.2. [3]. Let X be any non-empty set and let $\mathscr{D} \subset P(X \times X)$ be a subcollection of power set of $X \times X$. $\mathscr{D}$ is called a *local uniformity* on X if the following axioms are satisfied:

(i) $\Delta \subset D$ for each $D \in \mathscr{D}$ where $\Delta = \{(x,x) : x \in X\}$.

(ii) $D \in \mathscr{D} \Rightarrow D^{-1} \in \mathscr{D}$ for each $D \in \mathscr{D}$ where $D^{-1} = \{(x,y) : (y,x) \in D\}$

(iii) D_1, $D_2 \in \mathscr{D} \Rightarrow D_1 \cap D_2 \in \mathscr{D}$

(iv) $D \in \mathscr{D}$ and $D \subset E \subset X \times X \Rightarrow E \in \mathscr{D}$.

(v) for each $x \in X$ and for each $D \in \mathscr{D}$ there exists an $E \in \mathscr{D}$ such that $E \circ E [x] \subset D[x]$ where $E \circ E = \{(x, y) : \text{for some } z \in X,\ (x, z) \in E \text{ and } (z, y) \in E\}$.

The ordered pair $(X, \mathscr{D})$ is called a *locally uniform space.*

Also a subcollection $\mathscr{B} \subset P(X \times X)$ is called a *base* for some local uniformity on X if and only if $\mathscr{B}$ satisfies (i), (ii), (iii) and (v) above.

2. Covering locally uniform spaces

DEFINITION 2.1. A cover $\mathscr{U}$ of a locally uniform space $(X, \mathscr{D})$ is called a *locally uniform cover* if and only if it is refined by a cover of the form

$$\mathscr{U}_D = \{D[x] : x \in X\}$$

for some $D \in \mathscr{D}$.

THEOREM 2.2. *Let μ be the collection of all locally uniform covers of a locally uniform space $(X, \mathscr{D})$ then*

(i) *$\mathscr{U}_1, \mathscr{U}_2 \in \mu$ implies there exists $\mathscr{U}_3 \in \mu$ such that $\mathscr{U}_3 \ {}^{\Delta}\!\!< \mathscr{U}_1$ and $\mathscr{U}_3 \ {}^{\Delta}\!\!< \mathscr{U}_2$.*

(ii) *If $\mathscr{U} < \mathscr{V}$, $\mathscr{U} \in \mu$ then $\mathscr{V} \in \mu$.*

PROOF. Let μ and $(X, \mathscr{D})$ be as above. Let $\mathscr{U}_1$, $\mathscr{U}_2 \in \mu$. There exist, therefore, D_1, $D_2 \in \mathscr{D}$ such that $\mathscr{U}_{D_1} < \mathscr{U}_1$, $\mathscr{U}_{D_2} < \mathscr{U}_2$. Since $D_1, D_2 \in \mathscr{D}$, we have $D_1 \cap D_2 \in \mathscr{D}$. Now $(X, \mathscr{D})$ being locally uniform, for each $x \in X$ and $D_1 \cap D_2 \in \mathscr{D}$, there exists a symmetric $D \in \mathscr{D}$ such that $D \circ D\ [x] \subset D_1 \cap D_2\ [x]$. We claim $\mathscr{U}_D \ {}^{\Delta}\!\!< \mathscr{U}_{D_1}$ and $\mathscr{U}_D \ {}^{\Delta}\!\!< \mathscr{U}_{D_2}$. For it, it is sufficient to show that

$$\text{St}\ (x, \mathscr{U}_D) \subset D_1[x] \cap D_2[x].$$

Let,

$$y \in \text{St}\ (x, \mathscr{U}_D) = \bigcup_{\substack{x \in D[z] \\ z \in X}} D[z]$$

$\Rightarrow$ for some $z \in X$, $y \in D[z]$ and $x \in D[z]$

$\Rightarrow$ $(z, y) \in D$, $(z, x) \in D$

$\Rightarrow$ $(x, y) \in D \circ D$

$\Rightarrow$ $y \in D \circ D[x] \subset D_1[x] \cap D_2[x]$

Therefore,

$$\text{St}\ (x, \mathscr{U}_D) \subset D_1[x] \in \mathscr{U}_{D_1} < \mathscr{U}_1 \text{ implying that } \mathscr{U}_D \ {}^{\Delta}\!\!< \mathscr{U}_{D_1}.$$

Similarly we have $\mathscr{U}_D \stackrel{\Delta}{<} \mathscr{U}_{D_2}$. Also $\mathscr{U}_D$ is obviously in μ, let it be denoted by $\mathscr{U}_3$; and thus (i) is proved. (ii) part follows directly from the definition of locally uniform covers.

The converse of the above theorem is true. We state it as follows:

THEOREM 2.3. *Let μ be a family of covers of a non-empty set X satisfying (i) and (ii) of Theorem 2.1. Define for each*

$$D_{\mathscr{U}} = \bigcup \{u \times u : u \in \mathscr{U}\}.$$

Then the collection $\mathscr{B} = \{D_{\mathscr{U}} : \mathscr{U} \in \mu\}$ forms a base for a local uniformity on X whose local uniform covers are precisely the members of μ.

PROOF. Obviously each member of $\mathscr{B}$ contains Δ and each member is symmetric. Next let $D_{\mathscr{U}}$, $D_{\mathscr{V}} \in \mathscr{B}$ then $\mathscr{U}, \mathscr{V} \in \mu$ and hence there exists a $\mathscr{W} \in \mu$ such that $\mathscr{W} \stackrel{\Delta}{<} \mathscr{U}$, $\mathscr{W} \stackrel{\Delta}{<} \mathscr{V}$. Obviously, then $D_{\mathscr{W}} \subset D_{\mathscr{U}} \cap D_{\mathscr{V}}$. Finally we prove the local property. Let $D_{\mathscr{U}} \in \mathscr{B}$ and $x \in X$. $D_{\mathscr{U}} \in \mathscr{B}$ implies $\mathscr{U} \in \mathscr{B}$ and hence there exists $\mathscr{V} \in \mu$ such that $\mathscr{V} \stackrel{\Delta}{<} \mathscr{U}$. We claim that $D_{\mathscr{V}} \circ D_{\mathscr{V}}[x] \subset D_{\mathscr{U}}[x]$. Let $y \in D_{\mathscr{V}} \circ D_{\mathscr{V}}[x]$ then $(x, y) \in D_{\mathscr{V}} \circ D_{\mathscr{V}}$ and so for some $z \in X$, $(x, z) \in D_{\mathscr{V}}$ and $(z, y) \in D_{\mathscr{V}}$ which yield $x, z \in V$ for some $V \in \mathscr{V}$ and $z, y \in W$ for some $W \in \mathscr{V}$. Now we have St $(z, \mathscr{V}) \subset \mathscr{U}$ for some $u \in \mathscr{U}$. Hence $x, y \in \mathscr{U}$ implying that $(x, y) \in D_{\mathscr{U}}$ or $y \in D_{\mathscr{U}}[x]$ whence $D_{\mathscr{V}} \circ D_{\mathscr{V}}[x] \subset D_{\mathscr{U}}[x]$. Hence $\mathscr{B}$ is indeed a base for some local uniformity say $\mathscr{C}$ on X. In the final we show that the family of local uniform covers with respect to $\mathscr{C}$ is precisely μ. Let η denote the collection of all locally uniform covers with respect to $\mathscr{C}$. We show that $\eta = \mu$. Obviously $\mu \subset \eta$ by the assumption of μ, for, if $\mathscr{U} \in \mu$ then by (i) there exists $\mathscr{U}_1 \in \mu$ such that $\mathscr{U}_1 \stackrel{\Delta}{<} \mathscr{U}$. Then by definition $D_{\mathscr{U}_1} \in \mathscr{C}$ and $\{D_{\mathscr{U}_1}[x] : x \in X\} < \mathscr{U}$ whence $\mathscr{U} \in \eta$. To show the otherway inclusion, let $\mathscr{A} \in \eta$ then there exists $L \in \mathscr{C}$ such that $\mathscr{U}_L < \mathscr{A}$. Now $L \in \mathscr{C}$ implies there exists $D_{\mathscr{V}} \in \mathscr{B}$ such that $D_{\mathscr{V}} \subset L$. So $D_{\mathscr{V}}[x] \subset L[x]$ for each $x \in X$. But $D_{\mathscr{V}}[x] =$ St $(x, \mathscr{V})$ whence $\mathscr{V} < \{D_{\mathscr{V}}[x] : x \in X\} < \mathscr{A}$ and hence by (ii) we have $\mathscr{A} \in \mu$.

REMARK. Thus we see that the local uniform covers describe a local uniformity as well as its entourages. We call (X, μ) a *covering locally uniform space*. Rest of the theory of covering locally uniform spaces regarding defining of bases, subbases, subspaces and products can easily be done parallel to the theory of covering uniform spaces. We leave that as a simple exercise.

3. Compact sets and local uniform spaces

THEOREM 3.1. *In a locally uniform space* $(X, \mathscr{D})$ *let* A *be a compact set and* B *a closed set such that* $A \cap B = \phi$ *then there is an entourage* $u \in \mathscr{D}$ *such that* $u \cap A \times B = \phi$.

PROOF. Let $(X, \mathscr{D})$, A and B be as above. Since $A \cap B = \phi$, $A \subset X - B$, so for each $a \in A$ there is a $V_a \in \mathscr{D}$ such that $V_a[a] \subset X - B$. Now $a \in A$, $V_a \in \mathscr{D}$ there exists an $u_a \in \mathscr{D}$ such that $u_a u_a[a] \subset V_a[a]$. Now $\{u_a[a] : a \in A\}$ is an open cover of A which is compact and hence it has a finite subcover, say,

$$\{u_{a_i}[a_i] : i = 1, 2, \cdots\cdots\cdots, n\}.$$

Now put $u = \bigcap_{i=1}^{n} u_{a_i}$ then $u \in \mathscr{D}$ and it can be easily seen that $u \cap A \times B = \phi$.

COROLLARY 3.2. *In a locally uniform space* $(X, \mathscr{D})$ *let* A *be a closed set,* B *be a compact set such that* $A \cap B = \phi$ *then there exists a* $u \in \mathscr{D}$ *such that* $u[B] \cup A = \phi$.

THEOREM 3.3. *In a locally uniform space* $(X, \mathscr{D})$ *if* A *is a compact subset of* X *then the family* $\{u[A] : u \in \mathscr{D}\}$ *forms a fundamental system of neighbourhoods.*

PROOF. Let N be any open neighbourhood of A. Let $B = X - N$. B is then closed and $A \cap B = \phi$, hence, by corollary 3.2 there exists an entourage $u \in \mathscr{D}$ such that $u[A] \cap B = \phi$ implying that $A \subset u[A] \subset X - B = N$. Hence the theorem is proved.

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Institute of Advanced Studies,
Meerut University,
Meerut-250001
INDIA.

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