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# ASSOCIATED SEMIGROUPS BETWEEN TOPOLOGICAL SPACES

By Sung Kag Chang

## 0. Introduction

A few authors [3], [7] have defined several types of semigroup structures on topological spaces and studied the relations between semigroups and corresponding spaces.

In this paper, we start by defining another semigroup structure between two spaces, and call it an associated semigroup between two corresponding spaces.

Our purpose of this paper is to study the relations between associated semigroups and corresponding spaces.

## 1. Preliminaries

Throughout, $X$, $X'$, $Y$ and $Y'$ will denote all $T_1$-spaces.

DEFINITION 1.1. Let $X$ and $Y$ be spaces. A map $f$ from a subspace of $X$ onto a subspace of $Y$ is called a *c-map* between $X$ and $Y$ if it satisfies the following conditions;

(1) the domain of $f$ is closed in $X$.

(2) the range of $f$ is closed in $Y$.

(3) $f$ is continuous and closed.

DEFINITION 1.2. Let $X$ and $Y$ be spaces. For a given $c$-map $p$ from $Y$ into $X$, the multiplication $f*g$ of two $c$-maps $f$ and $g$ between $X$ and $Y$ is defined as the composition $f\circ p\circ g$.

Then the set $S(X,Y;p)$ of all $c$-maps between $X$ and $Y$ with the multiplication $*$ is a semigroup.

Hereafter, we call this semigroup $S(X,Y;p)$ as an *associated semigroup between X and Y.*

Conventionally, we have the following notations.

NOTATION 1.3. Let $S(X,Y;p)$ be an associated semigroup between $X$ and $Y$. For an element $f\in S(X,Y;p)$,

(1) Dom $(f)$ denotes the domain of $f$,

(2) Ran $(f)$ denotes the range of $f$,

(3) 0 denotes the empty map,

(4) $A_y$ denotes $f$ if Dom $(f)=A$ and Ran $(f)=\{y\}$.
In particular $x_y$ if Dom $(f)=\{x\}$ and Ran $(f)=\{y\}$,

(5) if $A$ is a subspace of $X$, $f|A$ denotes the restriction of $f$ to $A\cap$Dom $(f)$.

**2. Homomorphisms between associated semigroups.**

LEMMA 2.1. *Let $\Psi$ be any homomorphism from an associated semigroup $S(X, Y; p)$ into an associated semigroup $S(X', Y'; p')$ which maps all constants in $S(X, Y; p)$ into all constants in $S(X', Y'; p')$. Then for any constants $A_y$ and $B_y$ in $S(X, Y; p)$,*

$$Ran[\Psi(A_y)]=Ran[\Psi(B_y)].$$

PROOF. It is sufficient to show that for any constant $A_y$ in $S(X, Y; p)$, Ran $[\Psi(A_y)]=$Ran$[\Psi(X_y)]$. Since $\Psi$ maps all constants in $S(X,Y;p)$ into all constants in $S(X',Y';p')$, there exist constants $W_u$ and $A'_v(\neq 0)$ in $S(X',Y';p')$ such that $\Psi(X_y)=W_u$ and $\Psi(A_y)=A'_v$. Since $X_y*A_y=A_y$, $A'_v=\Psi(A_y)=\Psi(X_y*A_y)=\Psi(X_y)*\Psi(A_y)=W_u*A'_v$. Thus $A'_v=W_u*A'_v\neq 0$.
Therefore $u=v$, i.e. Ran $[\Psi(X_y)]=$Ran $[\Psi(A_y)]$.

LEMMA 2.2. *Let $\Psi$ be any homomorphism from $S(X, Y; p)$ into $S(X', Y'; p')$ which maps all constants in $S(X,Y;p)$ into all constants in $S(X',Y';p')$.*
*Then for any constants $f$ and $g$ in $S(X, Y; p)$,*

$$Dom\ [\Psi(f)]=Dom[\Psi(g)]\ \text{if}\ Dom\ (f)=Dom(g).$$

PROOF. Suppose Dom$(f)=$Dom$(g)=A$ and Ran$(f)=y_1$, Ran$(g)=y_2$. Then there exists constants $A'_{1y_1'}$ and $A'_{2y_2'}$ in $S(X',Y';p')$such that

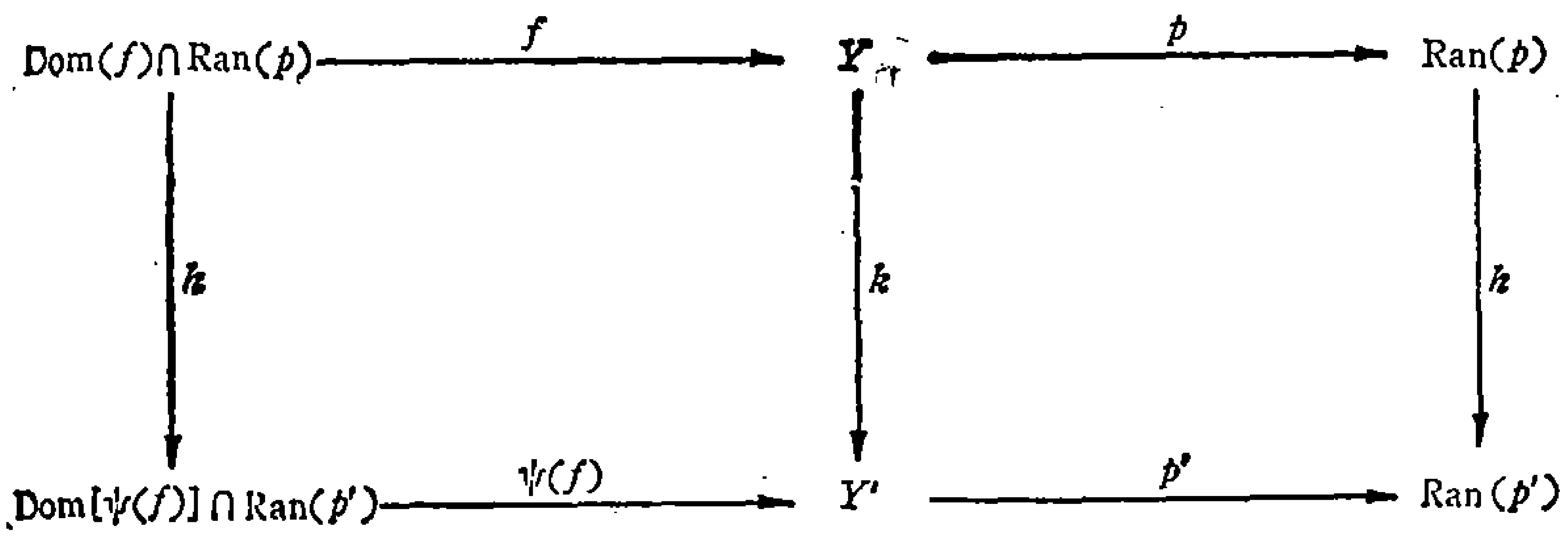
$$(A_{y_1})=A'_{1y_1'}\ \text{and}\ (A_{y_2})=A'_{2y_2'}.$$

Since $X_{y_1}*A_{y_1}=A_{y_1}=X_{y_1}*A_{y_2}$, by Lemma 2.1,

$$\Psi(X_{y_1}*A_{y_1})=W_{y_1'}*A'_{1y_1'}=A'_{1y_1'}\ \text{and}$$
$$\Psi(X_y*A_y)=W_{y_1'}*A_{2y_2'}=A'_{1y_1'}.$$

Thus $A'_1=A'_2$ i.e. Dom $[\Psi(A_{y_1})]=$Dom$[\Psi(A_{y_2})]$

THEOREM 2.3. *Let $\Psi$ be a homomorphism from an associated semigroup $S(X, Y; p)$ into an associated semigroup $S(X', Y'; p')$ which maps all constants in $S(X,Y;p)$ onto all constants in $S(X', Y'; p')$. Then there exist a continuous and closed map $h$ from $Ran(p)$ onto $Ran(p')$, and a map $k$ from $Y$ onto $Y'$ such that* *for any element $f \in S(X, Y; p)$, the following diagram commutes,*




*In this case, $h$ and $k$ are uniquely determined by $\Psi$.*

PROOF. By Lemma 2.1, we can define a map $k$ from $Y$ into $Y'$ as the following way, for each $y \in Y$, $k(y)=y'$ if for some constant $A_y \in S(X, Y; p)$, $\mathrm{Ran}[\Psi(A_y)]=y'$.

Then we have the follows. For any $y_1$ and $y_2$ in $p^{-1}(x)$, any given $x \in \mathrm{Ran}(p)$, $p'(k(y_1))=p'(k(y_2))$.

Suppose $k(y_1)=y_1'$ and $k(y_2)=y_2'$, $p'(y_1')=x_1'$ and $p'(y_2')=x_2'$.

Then there exist constants $f$ and $g$ in $S(X', Y'; p')$ such that $\Psi(x_{y_1})=f$, $\Psi(x_{y_2})=g$. By Lemma 2.1. $\mathrm{Ran}(f)=y_1'$ and $\mathrm{Ran}(g)=y_2'$, and by Lemma 2.2, $\mathrm{Dom}(f)=\mathrm{Dom}(g)$ say, $A'$.

Since $x_{y_1} * x_{y_2} = x_{y_1}$ and $x_{y_2} * x_{y_1} = x_{y_2}$,

$$A'_{y'} = \Psi(x_{y_1} * x_{y_2}) = \Psi(x_{y_1}) * \Psi(x_{y_2}) = A'_{y_1'} * A'_{y_2'} \text{ and } A'_{y_2'} = A'_{y_2'} * A'_{y_1'}.$$

Thus $x_1'=p'(y_1')$ and $x_2'=p'(y_2')$ are in $A'$. Because $\Psi$ maps all constants in $S(X, Y; p)$ onto all constants in $S(X', Y'; p')$ there exists a constant $A_{y_1}$ in $S(X, Y; p)$ such that $\Psi(A_{y_1})=x_1{}'_{y_1'}$.

Then $A_{y_1} * x_{y_1} \neq 0$ i.e. $p(y_1)=x \in A$. So $A_{y_1} * x_{y_2} = x_{y_1}$.

Therefore $\Psi(A_{y_1} * x_{y_2}) = \Psi(A_{y_1}) * \Psi(x_{y_2}) = x_1{}'_{y_1'} * A'_{y_2'} = A'_{y_1'}$.

So $x_2'=p'(y_2')=x_1'$.

Thus we can define a map $h$ from $\mathrm{Ran}(p)$ into $\mathrm{Ran}(p')$ as the following way, for each $x \in \mathrm{Ran}(p)$, $h(x)=p'(k(y))$ for any $y \in p^{-1}(x)$.

Then $h \circ p = p' \circ k$.

By the similar way, we can obtain that for any $f \in S(X,Y;p)$, $k \circ f = \Psi(f) \circ h$.

Thus the given diagram commutes.

Moreover the maps $h$ and $k$ are surjections, $h$ maps $\mathrm{Dom}(f) \cap \mathrm{Ran}(p)$ onto $\mathrm{Dom}[\Psi(f)] \cap \mathrm{Ran}(p')$ for any $f \in S(X, Y; p)$ from the fact that $\Psi$ maps all constants in $S(X, Y; p)$ onto all constants in $S(X', Y'; p')$.

From the above facts, we can easily see that $h$ and $k$ are uniquely determined by $\Psi$.

Finally, to show that $h$ is continuous and closed. Suppose $A'$ is any closed subset of $\mathrm{Ran}(p')$. Then there exists a constant $f'$ in $S(X', Y'; p')$ such that $\mathrm{Dom}(f')=A'$. Since $\Psi$ maps all constants in $S(X, Y; p)$ onto all constants in $S(X', Y'; p')$, there exists an element $f$ in $S(X, Y; p)$ such that $\Psi(f)=f'$.

Since the given diagram commutes, $h^{-1}(A')=\mathrm{Dom}(f)\cap\mathrm{Ran}(p)$. Thus $h$ is continuous. By the similar way, $h$ is closed. This proves the theorem.

From the above theorem, we have the following corollary.

COROLLARY 2.4. *In the Theorem 2.3, for any $f\in S(X, Y; p)$, $k|Ran(f)$ is continuous and closed, And if $\Psi$ is injective, then h and k are injective.*

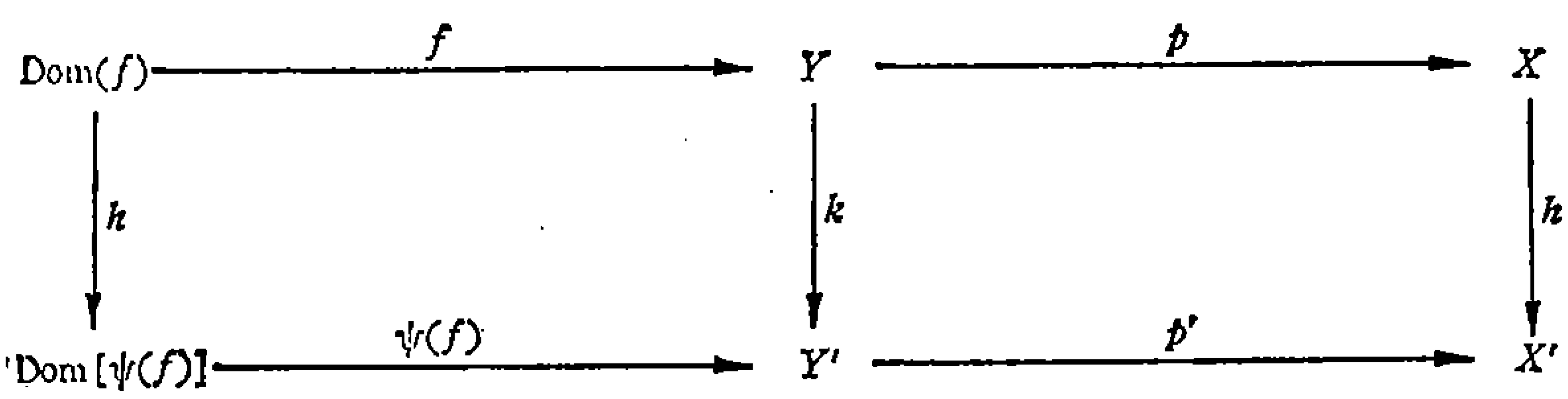
In the next part, we consider the case that in an associated semigroup $S(X, Y; p)$, the map $p$ is surjective, i.e. $\mathrm{Ran}(p)=X$. We denote such an associated semigroup by $S^*(X, Y; p)$.

Then we can obtain the following lemma.

LEMMA 2.5. *Let f be an element of $S^*(X, Y; p)$. Then f is a constant if and only if $f*g*f=f$ or 0 for any $g\in S^*(X, Y; p)$.*

Then we have the following theorem.

THEOREM 2.6. *A map $\Psi$ from $S^*(X, Y; p)$ onto $S^*(X', Y'; p')$ is an isomorphism if and only if there exist a homeomorphism h from X onto X' and a bijection k from Y onto Y' such that for any $f\in S^*(X, Y; p)$, the following diagram commutes;*




*and k maps Ran(f) homeomorphically onto Ran[$\Psi(f)$].*

PROOF. (Necessity) Let $\Psi$ be an isomorphism from $S^*(X, Y; p)$ onto $S^*(X', Y'; p')$. By the Theorem 2.3, and Corollary 2.4, it is sufficient to show that $\Psi$ maps all constants in $S^*(X, Y; p)$ onto all constants in $S^*(X', Y'; p')$. Suppose $f$ is any constant in $S^*(X, Y; p)$ and $g'$ is any element in $S^*(X', Y': p')$. Then there exists an element $g$ in $S^*(X, Y; p)$ such that

$$\Psi(g)=g'.$$

By Lemma 2.5, $f*g*f=f$ or 0. Thus $\Psi(f*g*f)=\Psi(f)*g'*\Psi(f)=\Psi(f)$ or 0. So, $\Psi(f)$ is a constant.

By the similar way, for any constant $f'$ in $S^*(X', Y'; p')$, $\Psi^{-1}(f')$ is a constant.

(Sufficiency) Suppose $h$ and $k$ are two maps which satisfy the necessary conditions. Then for any $f\in S^*(X, Y; p)$, the composition $k\circ f\circ h^{-1}$ is a $c$-map between $X'$ and $Y'$. Thus we can define a map $\Psi$ from $S^*(X, Y; p)$ into $S^*(X', Y'; p')$ by $\Psi(f)=k\circ f\circ h^{-1}$ for each $f\in S^*(X, Y; p)$. Then $\Psi$ is an isomorphism from $S^*(X, Y; p)$ onto $S^*(X', Y'; p')$. This proves the theorem,

REMARK. In the Theorem 2.6, the map $k$ need not be a homeomorphism. Next we have an example.

EXAMPLE. Let $Y_1$ be the set $\{1, 2, 3, 4\}$ with the discrete topology, and $Y_2$ the set $\{x\geq 5 | x\text{: natural numbers}\}$ with the cofinite topology. Assume $Y$ is the topological sum of $Y_1$ and $Y_2$ i.e. $Y=Y_1+Y_2$. Let $Y'$ be the set of all natural numbers with the discrete topology. Assume $X=X'$ is the set $\{0, 1, 2, 3, 4\}$ with the discrete topology. Then $X, Y$ and $Y'$ are $T_1$-spaces. Define a map $p$ from $Y$ onto $X$ and a map $p'$ from $Y'$ onto $X$ by

$$p(x)=p'(x)=x \text{ if } 1\leq x\leq 4,$$
$$p(x)=p'(x)=0 \text{ if } 5\leq x.$$

Then $p$ and $p'$ are continuous and closed. Define a map $h$ from $X$ onto $X$ as the identity map, and a map $k$ from $Y$ onto $Y'$ as also the identity map. For any $f\in S^*(X, Y; p)$, $k\circ f\circ h^{-1}=f$ is also a $c$-map between $X'$ and $Y'$. Thus we can define an isomorphism $\Psi$ from $S^*(X, Y; p)$ onto $S^*(X', Y'; p')$ by $\Psi(f)=k\circ f\circ h^{-1}$ for any $f\in S^*(X, Y; p)$. But $k$ is not a homeomorphism.

COROLLARY 2.7. *Assume there exists an element $f\in S^*(X, Y; p)$ such that $\mathrm{Ran}(f)=Y$. Then two associated semigroups $S^*(X, Y; p)$ and $S^*(X', Y'; p')$ are isomorphic if and only if there exist two homeomorphisms $h$ and $k$ from $X$ onto $X'$ and from $Y$ onto $Y'$ respectively such that $h\circ p=p'\circ k$.*

Let $p$ be a map from $Y$ into $X$. $\Pi(p)$ denotes the set of all $p^{-1}(x)$ for $x\in\mathrm{Ran}(p)$.

Then we have the following corollary.

COROLLARY 2.8. *In $S^*(X, Y; p)$ and $S^*(X', Y'; p')$, assume $\Pi(p)$ and $\Pi(p')$*

*are nbd-finite families, and there exists an element $f \in S^*(X, Y; p)$ such that Ran $(f)=Y$. Then two associated semigroups $S^*(X, Y; p)$ and $S^*(X', Y'; p')$ are isomorphic if and only if there exists a bijection $\phi$ from $\Pi(p)$ onto $\Pi(p')$ such that $p^{-1}(x)$ and $\phi(p^{-1}(x))$ are homeomorphic for each $p^{-1}(x) \in \Pi(p)$.*

PROOF. By the Corollary 2.7, together with [2], we can easily prove the corollary.

Kyungpook University

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