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# ON CERTAIN REFLECTIONS IN TOPOLOGICAL ALGEBRAIC SITUATIONS

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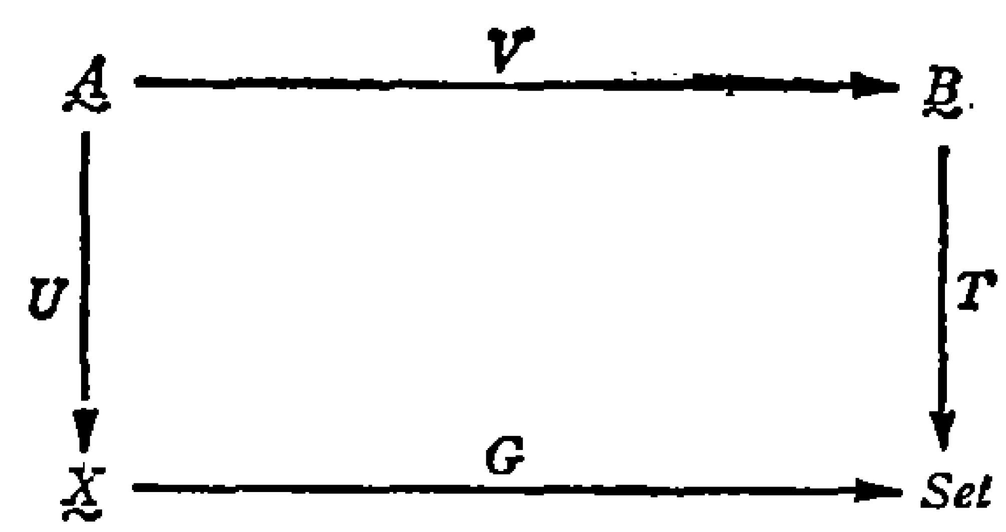
**Abstract**

Consider the following commutative square of categories and functors where $T$ and $U$ are (regular epi, monosource) topological and fibre small, $G$ is algebraic and the following two conditions hold: (i) if $h$: $UA \to UB$ and $g$: $VA \to VB$ are morphisms such that $Gh=Tg$, there exists a morphism $f$: $A \to B$ such that $Uf=h$ and $Vf=g$; (ii) $U$-initial monosources are carried into $T$-initial monosources by $V$.




Such a square is called a Topological Algebraic Situation ($TAS$).

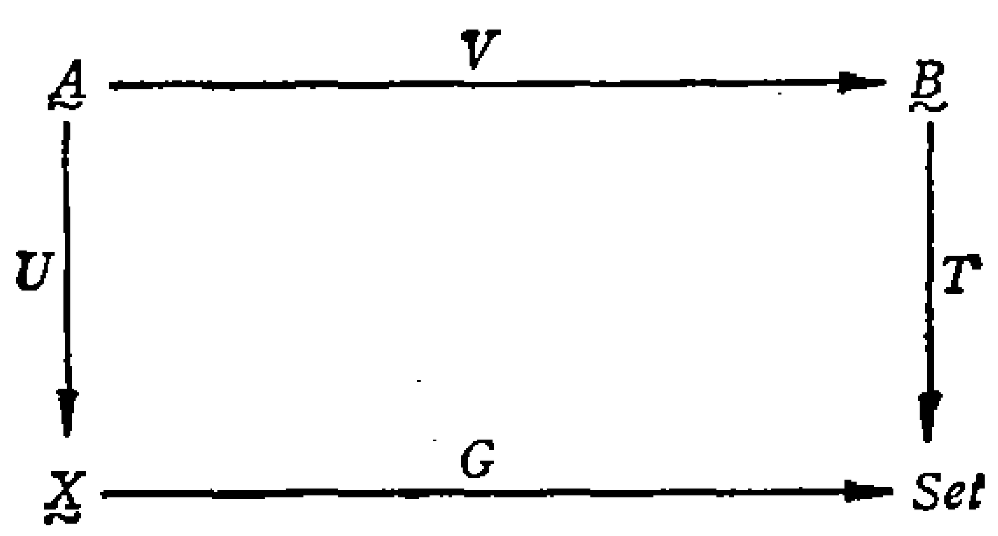
In a previous work the author showed how much of the classical theory of topological algebra is recaptured with in this axiomatic setting. In particular, it follows that if $\underset{\sim}{B}'$ is a "surjective"-reflective subcategory of $\underset{\sim}{B}$, the full subcategory $\underset{\sim}{A}'$ of $\underset{\sim}{A}$, consisting of all objects $A$ with $VA$ a $\underset{\sim}{B}'$-object, is "surjective"-reflective in $\underset{\sim}{A}$ and with $U'$, $V'$ and $T'$ the obvious restrictions, $GU'=T'V'$ is a $TAS$.

In this note it is shown that if $\underset{\sim}{A}$ has a suitable factorization property for its morphisms, and if $\underset{\sim}{B}'$ is merely epi-reflective in $\underset{\sim}{B}$, then $\underset{\sim}{A}'$ is reflective in $\underset{\sim}{A}$. In contrast to the "surjective" case, $GU'=T'V'$ need not be a $TAS$. However, it is shown that $U'$, $V'$, $T'$ and $GU'$ are well behaved functors in that they are topologically algebraic in the sense of Y.H. Hong.

Examples of this epi-reflection case are the Bohr and zero-dimensional compactifications of a Hausdorff group (or semigroup). By considering the categories of pairwise Tychonoff bitopological spaces and pairwise Hansdorff bitopological groups, Bohr and pairwise zero-dimensional pairwise compactifications are obtained. This is seen as further justification of the axiomatic approach being developed.

## 0. Introduction

In this note we continue the study of an axiomatic approach to categories of topological algebras begun in [3]. Consider the commuting square of categories and functors




where *Set* is the category of sets, $(\underset{\sim}{X}, G)$ is an algebraic situation, both of $U$ and $T$ are (regular, epi, monosource) topological functors having small fibres. We call this square a Topological Algebraic Situation (*TAS*) if the following two conditions hold:

(i) if $h$: $UA \to UB$ and $g$: $VA \to VB$ are morphisms with $Bh = Tg$, there exists a morphism $f$: $A \to B$ such that $Uf = h$ and $Vf = g$

(ii) $U$-initial monosources are carried into $T$-initial monosources by $V$.

The assumption that $T$ be (regular epi, monosource) topological and fibre small assures that $\underset{\sim}{B}$ is a reasonable analogue of a "topological" category. See Herrlich [6] and Nel [10].

Fibre smallness for $U$ is viewed as reasonable since for any algebra there should be at most a set of topologies making the algebra a topological algebra. The (regular epi, monosource) topological requirement is simply a categorical interpretation of the following result for Hausdorff groups.

If $(G_i)_I$ is a family of Hausdorff groups, $G$ is a group, and $f_i$: $G \to G_i$ is a group homomorphism for each $i \in I$. Then the coarsest topology making each $f_i$ continuous is a group topology on $G$. Moreover, if $H$ is a Hausdorff group and $h$: $H \to G$ is a group homomorphism, $h$ is continuous when $G$ is endowed with the coarse topology determined by the family $(f_i)_I$ if and only if $f_i h$ is continuous for each $i \in I$. However, $G$ need not be a Hausdorff group unless the family $(f_i)_I$ is point separating (monosource).

Condition (i) is essentially a "fullness" condition. Loosely speaking, (i) means any function simultaneously a homomorphism of the underlying algebras and continuous with respect to the underlying topologies is a morphism of the topological algebras.

Condition (ii) means an "embedding" of topological algebras is simply an embedding of the underlying spaces. Condition (ii) is a powerful condition in that it is closely related to the existence of "the free topological algebra over a topological space" (see Theorem 5.3 of [3]).

If $(\underset{\sim}{X}, G)$ is any (finitary or infinitary) equationally defined class of universal algebras of fixed type, $\underset{\sim}{B}$ is the category of topological spaces, ***Top***, and an object in $\underset{\sim}{A}$ is an object from $\underset{\sim}{X}$ endowed with a topology making each of the defining operations continuous in the usual manner, and a morphism between objects in $\underset{\sim}{A}$ is simply a homomorphism of the underlying algebras which is continuous on the underlying spaces, then with $U$, $V$, $G$ and $T$ the obvious forgetful functors, we have a *TAS*.

If $(\underset{\sim}{X}, G)$ is the category of $\mathcal{C}$-algebras with forgetful functor $G$, $\underset{\sim}{B}$ is the category of Hausdorff $k$-spaces, and $\underset{\sim}{A}$ the category of $k$-$\mathcal{C}$-algebras (see [2]), then, again, with $U$, $V$ and $T$ the obvious forgetful functors we have a *TAS*.

The categories $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{X}$ are complete, cocomplete, well powered and regular epi-cowell powered. Each of the functors $U$, $V$, $G$ and $T$ has a left adjoint and hence each is limit preserving; each is faithful, hence monomorphism reflecting. Much of the theory of classical topological algebra is recaptured with a *TAS* and indeed the main thrust of [3] is to show this.

If $\underset{\sim}{B}'$ and $\underset{\sim}{X}'$ are "surjective"-reflective subcategories of $\underset{\sim}{B}$ and $\underset{\sim}{X}$ respectively, and $\underset{\sim}{A}'$ is the full subcategory of $\underset{\sim}{A}$ consisting of all objects $A$ with $UA$ an $\underset{\sim}{X}'$-object and $VA$ a $\underset{\sim}{B}'$-object, then $\underset{\sim}{A}'$ is "surjective"-reflective in $\underset{\sim}{A}$. Moreover, if $U'$, $V'$, $T'$ and $G'$ are the obvious restrictions of $U$, $V$, $T$ and $G$, then $G'U'=T'V'$ is a *TAS* [3].

In this paper we consider an epi-reflective subcategory $\underset{\sim}{B}''$ of $\underset{\sim}{B}$ and the full subcategory $\underset{\sim}{A}''$ of $\underset{\sim}{A}$ consisting of all objects $A$ with $VA$ a $\underset{\sim}{B}''$-object. It follows that if $\underset{\sim}{A}$ has a suitable factorization property for its morphisms, then $\underset{\sim}{A}''$ is an epi-reflective subcategory of $\underset{\sim}{A}$. But in contrast to the "surjective"-reflection case above, $GU''=T''V''$ need not be a *TAS*. However, these obvious restriction functors $U''$, $V''$, $T''$ and $GU''$ are well behaved as they are topologically algebraic in the sense of Y.H. Hong [9] (see also S.S. Hong [8]).

This epi-reflection case yields the Bohr and zero-dimensional compactifications of a Hausdorff group (or semigroup) as special cases. We obtain further new examples by considering pairwise Hausdorff bitopological spaces and obtaining a Bohr pairwise compactification and a pairwise zero-dimensional compactification. This is seen as further justification of the axiomatic approach being developed.

The author wishes to acknowledge helpful discussions with G. C. L. Brümmer concerning bitopological spaces and bitopological groups.

## 1. Preliminaries

For the sake of brevity we refer the reader to the fundamental work by H. Herrlich [6] and to Section 1 of the important paper [10] by L. D. Nel for the notions of initiality and $(\underset{\sim}{E}, \underset{\sim}{M})$ topological functor. For the theory of algebraic categories and algebraic functors as well as for categorical terminology not expressly defined herein, we refer to the text by H. Herrlich and G. E. Strecker [7].

It will be convenient to adopt some notation. The class of all epics (respectively extremal monics) in a category $\underset{\sim}{A}$ will be denoted by $Epi_{\underset{\sim}{A}}$ (resp. $Ext\ Mono_{\underset{\sim}{A}}$). If $V$: $\underset{\sim}{A} \to \underset{\sim}{B}$ is a functor, the class of all morphisms $f$ in $\underset{\sim}{A}$ with $Vf$ in $Epi_{\underset{\sim}{B}}$ (resp. $Ext\ Mono_{\underset{\sim}{B}}$) will be denoted by $Epi_V$ (resp., $Ext\ Mono_V$). If $\underset{\sim}{M}$ is a class of sources in $\underset{\sim}{B}$, denote the class of all $V$-initial sources $(A, f_i)_I$ in $\underset{\sim}{A}$ with the property that $(VA, Vf_i)_I \in \underset{\sim}{M}$ by $\underset{\sim}{M}_V$.

All subcategories considered are full and replete (=isomorphism closed).

PROPOSITION 1.1 *Given a TAS as in Section* 1,

(a) $Epi_V \subset Epi_{\underset{\sim}{A}}$.

(b) *If* $Epi_V = Epi_{\underset{\sim}{A}}$, *then* $Ext\ Mono_{\underset{\sim}{A}} \subset Ext\ Mono_V$.

(c) $\underset{\sim}{A}$ *is an* $(Epi_V, Ext\ Mono_V)$ *category if and only if* $\underset{\sim}{A}$ *is* $(Epi_V, Ext\ Mono_V)$-*factorizable.*

PROOF. (a) $V$ is faithful.

(b) If $Vf$ is an extremal monic in $\underset{\sim}{B}$, then $f$ is monic in $\underset{\sim}{A}$. Let $f=me$ be the (epi, extremal mono) factorization of $f$. Then $Vf=VmVe$. The hypothesis implies $Ve$ is an isomorphism and $V$ reflects isomorphisms [3].

(c) If $\underset{\sim}{A}$ is $(Epi_V$, Ext $Mono_V)$ factorizable, we need only show $\underset{\sim}{A}$ has the $(Epi_V, Ext\ Mono_V)$ diagonalization property to see $\underset{\sim}{A}$ is an $(Epi_V, Ext\ Mono_V)$ category. Let $e:X \to Y$ be a morphism in $\underset{\sim}{A}$ such that $Ve$ is epic in $\underset{\sim}{B}$. Let $m$: $Z \to W$ be a morphism in $\underset{\sim}{A}$ such that $Vm$ is an extremal monic in $\underset{\sim}{B}$. Assume there are morphisms $f$ and $g$ in $\underset{\sim}{A}$ such that $ge=mf$. It follows that there exists a morphism $k$: $VY\to VZ$ such that $Vmk=Vg$ and $kVe=Vf$. Since $(GUm)(Tk)=GUg$ and monics are $G$-initial, there exists a morphism $k'$ : $UY\to UZ$ such that $Umk'=Ug$ and $k'Ue=Uf$. Now apply condition (i).

Note this argument can be extended to show $\underset{\sim}{A}$ is an $(Epi_V, (Ext\ Monosource)_V)$ category if and only if $\underset{\sim}{A}$ is $(Epi_V, (Ext\ Monosource)_V)$ factorizable.

DEFINITIONS 1.2. Let $V:\underset{\sim}{A}\to\underset{\sim}{B}$ be a functor. A $\underset{\sim}{B}$-morphism $g:B\to VA$ is said to *V-generate* $A$ if for any pair of $\underset{\sim}{A}$-morphisms $A \overset{r}{\underset{s}{\rightrightarrows}} A'$, $(V_r)g=(V_s)g$ implies $r=s$.

The functor $V$ is called ***topologically algebraic*** if for each family $(A_i)_I$ of $\underset{\sim}{A}$-objects and each source $(B, B\xrightarrow{b_i}VA_i)_I$ in $\underset{\sim}{B}$, there exists a $V$-initial source $(A, A\xrightarrow{a_i}A_i)_I$ and $\underset{\sim}{B}$-morphism $b$: $B\to VA$ which $V$-generates $A$ and such that $(V_{a_i})b=b_i$ for each $i\in I$.

Topologically algebraic functors, first defined by H. Hong [9], generalize the topological functors of Herrlich [6] and most forgetful functors from categories of topological algebras are topologically algebraic, as is to be expected. S.S. Hong [8] has shown that topologically algebraic functors are precisely those functors having left adjoints and such that the domain categories are (epi, initial) factorizable.

THEOREM 1.3. (S.S Hong) *A functor $V$: $\underset{\sim}{A}\to\underset{\sim}{B}$ is topologically algebraic if and only if $V$ has a left adjoint and $\underset{\sim}{A}$ is (epi, $V$-initial)-fatorizable. Moreover, if $V$ has a left adjoint and $\underset{\sim}{A}$ is ($\underset{\sim}{E}$, $V$-initial)-factorizable for any class of epics $\underset{\sim}{E}$, then $V$ is topologically algebraic.*

## 2. The Epireflection situation

Consider the $TAS$ $GU=TV$ as defined in Section 0. Let $\underset{\sim}{B}''$ be an epi-reflective subcategory of $\underset{\sim}{B}$ and let $\underset{\sim}{A}''$ be the full subcategory of $\underset{\sim}{A}$ consisting of all objects $A$ with $VA$ a $B''$-object.

THEOREM 2.1. *If $\underset{\sim}{A}$ is $(Epi_V, Ext\ Mono_V)$ factorizable, then $\underset{\sim}{A}''$ is $Epi_V$-reflective in $\underset{\sim}{A}$.*

PROOF. From Proposition 1.1, $\underset{\sim}{A}$ is an $(Epi_V, Ext\ Mono_V)$ category. Thus a subcategory is $Epi_V$-reflective if and only if it is closed under products and $Ext\ Mono_V$-subobjects. Since $\underset{\sim}{B}''$ is closed under products and $V$ is limit preserving, $\underset{\sim}{A}''$ is closed under products. $\underset{\sim}{A}''$ is closed under $Ext\ Mono_V$-subjects since

$\underline{B}''$ is closed under extremal subobjects.

There are obvious restrictions $U''$, $V''$ and $T''$ of the functors $U$, $V$ and $T$ so that $GU''=T''V''$.

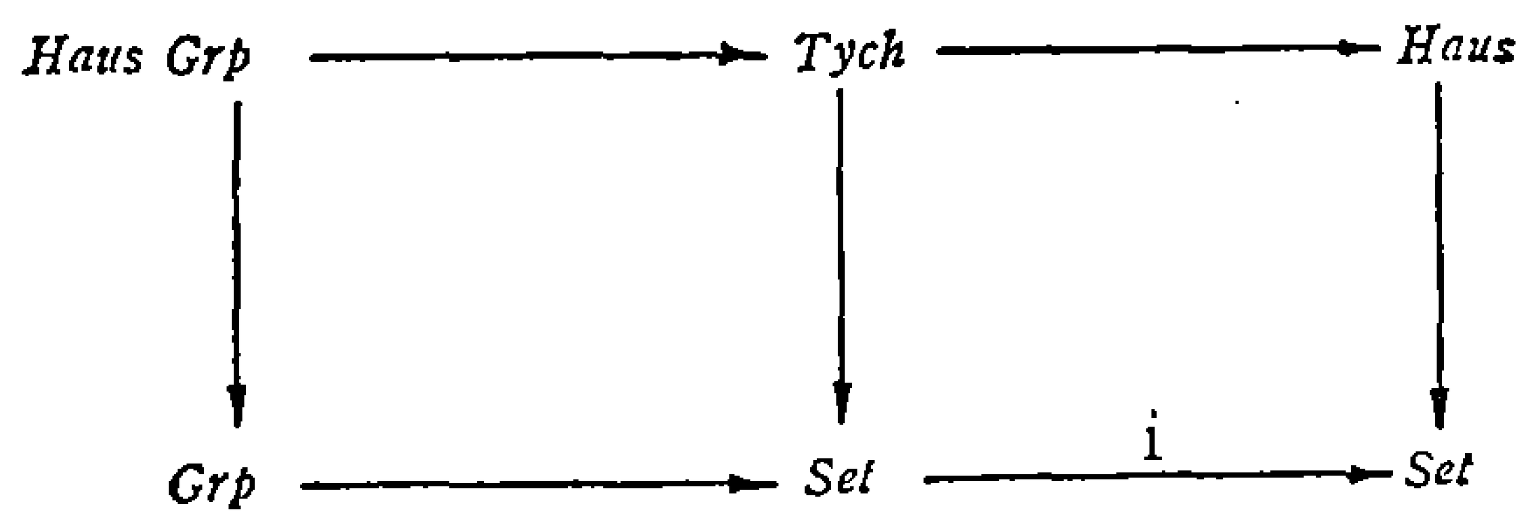
In contrast to the reflection theorem for surjective-reflective subcategories obtained in [3], $GU''=T''V''$ need not be a *TAS*. Take, for example, $\underline{B}''$ to be the category of compact Hausdorff spaces, $\underline{A}''$ the category of compact Hausdorff groups, and $\underline{X}$ the category of groups. Then neither $U''$ or $T''$ is (regular epi, monosource) topological.

On the other hand, the functors $U''$, $V''$ and $T''$ are well behaved as the next theorem shows. This theorem subsumes Corollary 2.6 of [8].

THEOREM 2.2. *Each of the functors $U''$, $V''$, $T''$ and $GU''$ is topologically algebraic.*

PROOF. Let $E''$ be the full inclusion of $\underline{A}''$ into $\underline{A}$. Since $\underline{A}''$ is $Epi_V$-reflective in $\underline{A}$, $E''$ is $(Epi_V, (Ext\ Monosource)_V)$ topological and hence $\underline{A}''$ is an $(Epi_{VE''}, (Ext\ Monosource)_{VE''})$ category. Faithfulness implies $Epi_{VE''} \subset Epi_{\underline{A}''}$. Every monosource is $V$-initial. Consequently extremal monources are $VE''$-initial. Hence from Theorem 1.3, $VE''$ is topologically algebraic. Further, the functor $V''$ must then be topologically algebraic. Similar reasoning yields $T''$ and $U''$ topologically algebraic. Since algebraic functors are topologically algebraic and composition of topologically algebraic functors are again so, it follows that $GU''$ is also topologically algebraic.

EXAMPLES 2.3. Let *Haus Grp* denote the category of Hausdorff groups, *Tych* the category of Tychonoff spaces, *Haus* the category of Hausdorff spaces, and *Grp* the category of groups. Then, in the following diagram each of the inner squares (and the outer perimeter) is a *TAS*. All functors are "forgetful".



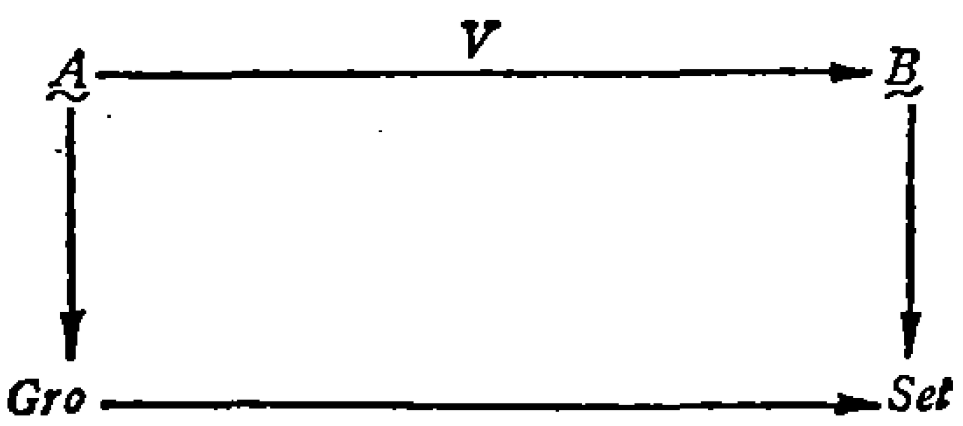

Let $\underline{B}'$ be the category of all compact Hausdorff spaces and let $\underline{B}''$ be the category of all zero-dimensional compact Hausdorff spaces. We may view $\underline{B}'$ and $\underline{B}''$ as full epi-reflective subcategories of either *Tych* or *Haus*. We obtain, in this setting, the Bohr-compactification and a zero-dimensional compactification of a Hausdorff group as examples.

Perhaps more interestingly, we obtain an analogue to the Bohr compactification for pairwise Hausdorff bitopological groups.

DEFINITION 2.4. A triple $((G, m), P, Q)$ is a *bitopological group* if $(G, m)$ is a group with operation $m:G\times G\to G$ and $P$ and $Q$ are topologies each making $m$ continuous in the usual manner and such that inversion $(-)^{-1}$: $(G, P)\to(G, Q)$ is a homeomorphism.

T. Bîrsan [0] and, independently, G.C.L. Brümmer [1] have shown that given such a bitopological group $((G, m), P, Q)$, $((G, m), P\vee Q)$ is a topological group. Moreover, if $(G, P, Q)$ is pairwise Hausdorff, then it is pairwise Tychonoff.

It follows easily that if $\underset{\sim}{A}$ is the category of pairwise Hausdorff groups, $\underset{\sim}{B}$ is the category of pairwise Tychonoff spaces, then with all functors being "forgetful" we have a *TAS* and $\underset{\sim}{A}$ has the $(Epi_V, \ Ext\ Mono_V)$-factorization property.




Considering the pairwise compact and pairwise zero-dimensional pairwise compact full subcategories of the category of pairwise Tychonoff spaces, we obtain Bohr and pairwise zero-dimensional pairwise compactifications of a pairwise Hausdorff bitopoogical group as examples. For definitions of these "pairwise" notions we refer the reader to Halpin [5] and Salbany [11].

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