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ON TOPOLOGIES WITH IDENTICAL DENSE SETS

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1. Introduction and background

We shall term two topologies $\mathscr{T}$ and $\mathscr{U}$ on a set X to be equivalent (and write $\mathscr{T}\equiv\mathscr{U}$) iff $(X, \mathscr{T})$ and $(X, \mathscr{U})$ have identical dense sets.

It is the intent of this paper to study some of the stable properties of congruence of topologies and to investigate extremal members of $[\mathscr{T}]$, the equivalence class determined by $\mathscr{T}$.

We will make frequent use of the following concepts:

DEFINITION 1.1. A topology $\mathscr{T}$ on a set X is a *D-topology* iff every nonempty open set is dense in X (see [1]).

DEFINITION 1.2. A topology $\mathscr{T}$ is an *S-topology* iff every superset of a nonempty open set is open (see [3]).

DEFINITION 1.3. Suppose $(X, \mathscr{T})$ is a topological space and $A\subset X$. Then $\mathscr{T}[A]$ denotes the supremum of $\mathscr{T}$ and $\{\phi, A, X\}$ and is called the *simple extension* of $\mathscr{T}$ by A (see [2]).

DEFINITION 1.4. A set A in a space $(X, \mathscr{T})$ is *semi-open* iff $A\subset c$ Int A, c denoting closure and Int denoting interior. $S(\mathscr{T})$ denotes the set of all semi-open sets (see [4]).

In §2, several characterizations of equivalence are given. It is shown that the following properties are invariant relative to equivalence: indiscreteness, discreteness, D-topology, separability, resolvability, first category, Baire space.

In §3, we show that congruence behaves smoothly relative to product and sum spaces; sufficient conditions are given for $Y\cap\mathscr{T}\equiv Y\cap\mathscr{U}$ when $T\equiv\mathscr{U}$ and $Y\subset X$. If $(X^*, \mathscr{T}^*)$ and $(X^*, \mathscr{U}^*)$ are the one-point compactifications of $(X, \mathscr{T})$ and $(X, \mathscr{U})$, then $\mathscr{T}^*\equiv\mathscr{U}^*$ iff $\mathscr{T}\equiv\mathscr{U}$ and $\mathscr{T}$ and $\mathscr{U}$ are both compact or both noncompact.

§4, 5, 6, 7 treat extremal members of $[\mathscr{T}]$ and §8 consists of examples and counterexamples.

Finally, c_t and c_u denote closure operators relative to $\mathscr{T}$ and $\mathscr{U}$ and Int_t and Int_u denote the respective interior operators. $\mathscr{C}$ denotes complementation.

2. General properties

We now give several characterizations of equivalence.

THEOREM 2.1. *Let* $(X, \mathscr{T}, \mathscr{U})$ *be a bitopological space. The following are equivalent*: (i) $\mathscr{T} \equiv \mathscr{U}$ (ii) $\phi \neq O \in \mathscr{T}$ *implies there exists a* $U \in \mathscr{U}$ *such that* $\phi \neq U \subset O$ *and* $\phi \neq U^* \in \mathscr{U}$ *implies there exists an* $O^* \in \mathscr{T}$ *such that* $\phi \neq O^* \subset U^*$ (iii) *for each* $A \subset X$, $Int_t\ A \neq \phi$ *iff* $Int_u\ A \neq \phi$ (iv) *for each* $A \subset X$, $Int_t\ c_u A \subset c_t A$ *and* $Int_u\ c_t A \subset c_u A$.

PROOF. (i)→(ii) Let $\phi \neq O \in \mathscr{T}$; then $\mathscr{C}O$ is not $\mathscr{T}$-dense and hence not $\mathscr{U}$-dense. Thus, $c_u\ \mathscr{C}O \neq X$ and $\mathscr{C}c_u\mathscr{C}O \neq \phi$. Take $U = \mathscr{C}c_u\mathscr{C}O$.

(ii)→(iii) is clear

(iii)→(iv) Suppose $\mathrm{Int}_t\ c_u A \not\subset c_t A$; then $\mathrm{Int}_t c_u A \cap \mathscr{C}c_t A \neq \phi$ and hence $\mathrm{Int}_t(c_u A \cap \mathscr{C}A) \neq \phi$. (iii) implies that $\mathrm{Int}_u(c_u A \cap \mathscr{C}A) \neq \phi$. But $\mathrm{Int}_u(c_u A \cap \mathscr{C}A) = \mathrm{Int}_u c_u A \cap \mathscr{C}c_u A = \phi$, a contradiction.

(iv)→(i) Let A be $\mathscr{T}$-dense. Then $c_u A \supset \mathrm{Int}_u\ c_t A = \mathrm{Int}_u\ X = X$. Thus A is $\mathscr{U}$-dense.

COROLLARY 2.2. *Let* $(X, \mathscr{T}, \mathscr{U})$ *be a bitopological space. Then* $\mathscr{T} \equiv \mathscr{U}$ *iff* $O \in \mathscr{T}$ *implies there exists a* $U \in \mathscr{U}$ *such that* $U \subset O$ *and* $c_t U = c_t O$ *and* $U^* \in \mathscr{U}$ *implies there exists an* $O^* \in \mathscr{T}$ *such that* $O^* \subset U^*$ *and* $c_u O^* = c_u U^*$.

PROOF. The sufficiency follows from (ii) of theorem 2.1. To show the necessity, let $O \in \mathscr{T}$ and $U = \mathrm{Int}_u O$; clearly $c_t U \subset c_t O$. It suffices to show then that $O \subset c_t U$; suppose however that $O \cap \mathscr{C}c_t U \neq \phi$. By (iii) of theorem 2.1, $\mathrm{Int}_u (O \cap \mathscr{C}c_t U) \neq \phi$ and hence $\mathrm{Int}_u O \cap \mathscr{C}c_t U \neq \phi$. But $\mathrm{Int}_u O \cap \mathscr{C}c_t U = U \cap \mathscr{C}c_t U = \phi$, a contradiction.

We now list some properties which are invariant relative to equivalence in

THEOREM 2.3. *Let* $(X, \mathscr{T}, \mathscr{U})$ *be a bitopological space and* $\mathscr{T} \equiv \mathscr{U}$. *Then* (i) $\mathscr{T}$ *is indiscrete iff* $\mathscr{U}$ *is indiscrete* (ii) $\mathscr{T}$ *is discrete iff* $\mathscr{U}$ *is discrete* (iii) $\mathscr{T}$ *is a D-topology iff* $\mathscr{U}$ *is a D-topology* (*see definition* 1.1) (iv) $\mathscr{T}$ *is separable iff* $\mathscr{U}$ *is separable* (v) $\mathscr{T}$ *is resolvable iff* $\mathscr{U}$ *is resolvable* (*a space is resolvable iff a subset and its complement are both dense*) (vi) *if* $A \subset X$, *then* A *is* $\mathscr{T}$*-nowhere* *dense iff A is* $\mathscr{U}$*-nowhere dense* (vii) $\mathscr{T}$ *is of first category iff* $\mathscr{U}$ *is of first category* (viii) $(X, \mathscr{T})$ *is a Baire space iff* $(X, \mathscr{U})$ *is a Baire space.*

PROOF. (i) Let $\mathscr{T}$ be indiscrete and $\phi \neq U \in \mathscr{U}$. By (ii) of theorem 2.1, there exists an $O \in \mathscr{T}$ such that $\phi \neq O \subset U$. Then $O = X$ and hence $U = X$.

(ii) Let $\mathscr{T}$ be discrete and $x \in X$. Then $\{x\} \in \mathscr{T}$ and by (ii) of theorem 2.1, $\{x\} \in \mathscr{U}$. Thus $\mathscr{U}$ is discrete.

(iii) Let $\mathscr{T}$ be a D-topology and $\phi \neq U \in \mathscr{U}$. By corollary 2.2, there exists an $O \in \mathscr{T}$ such that $O \subset U$ and $c_u O = c_u U$. Then $O \neq \phi$ and since $\mathscr{T}$ is a D-topology, O is $\mathscr{T}$-dense and hence $\mathscr{U}$-dense. Thus U is $\mathscr{U}$-dense and $\mathscr{U}$ is a D-topology.

(iv) and (v) are obvious.

(vi) Let $A \subset X$ and let A be $\mathscr{T}$-nowhere dense; then $\mathrm{Int}_t c_t A = \phi$. By (iv) of theorem 2.1, $\mathrm{Int}_t c_u A \subset \mathrm{Int}_t c_t A = \phi$ and hence $\mathrm{Int}_t c_u A = \phi$. By (iii) of theorem 2.1, $\mathrm{Int}_u c_u A = \phi$ and A is $\mathscr{U}$-nowhere dense.

(vii) follows from (vi).

(viii) Let $(X, \mathscr{T})$ be a Baire space and suppose $U_n \in \mathscr{U}$, U_n is $\mathscr{U}$-dense for each $n \geqq 1$. By corollary 2.2, there exist $O_n \in \mathscr{T}$ such that $O_n \subset U_n$ and $c_u O_n = c_u U_n = X$ for each $n \geqq 1$. Hence $c_u O_n = X$ and thus $c_t O_n = X$. It follows then that $X = c_t \cap O_n \subset c_t \cap U_n \subset X$ and $\cap U_n$ is $\mathscr{T}$-dense; hence $\cap U_n$ is $\mathscr{U}$-dense.

THEOREM 2.4. *Let* $(X, \mathscr{T}, \mathscr{U})$ *is a bitopological space,* $\mathscr{T} \equiv \mathscr{U}$ *and* $\mathscr{U} \subset \mathscr{T}$. *If* $\mathscr{T}$ *is regularly open* ($O = Int_t c_t O$ *for each* $O \in \mathscr{T}$), *then* $\mathscr{U}$ *is regularly open.*

PROOF. Let $U \in \mathscr{U}$; it suffices to show that $U \supset \mathrm{Int}_u c_u U$. By corollary 2.2, there exists an $O \in \mathscr{T}$ such that $O \subset U$ and $c_u O = c_u U$. Now $\mathrm{Int}_t c_t O = O \subset U$. Hence $U \supset \mathrm{Int}_t c_t O \supset \mathrm{Int}_t c_u O$ (by (iv) of theorem 2.1) $\supset \mathrm{Int}_u c_u O$ (since $\mathscr{U} \subset \mathscr{T}$) $= \mathrm{Int}_u c_u U$.

Frequent use is made of

LEMMA 2.5. *Let* $\mathscr{T} \subset \mathscr{T}^* \subset \mathscr{U}$ *be topologies on X and* $\mathscr{T} \equiv \mathscr{U}$. *Then* $\mathscr{T}^* \equiv \mathscr{U}$.

PROOF. Apply (ii) of theorem 2.1.

3. Subspaces, products, sums

Equivalence is not invariant relative to subspace (see example 8.4). However, we have

LEMMA 3.1. *Let* $(X, \mathscr{T}, \mathscr{U})$ *be a bitopological space,* $\mathscr{T} \equiv \mathscr{U}$ *and* $Y \subset X$.
(1) *If Y is* $\mathscr{T}$*-dense (and hence* $\mathscr{U}$*-dense) in X, then* $Y \cap \mathscr{T} \equiv Y \cap \mathscr{U}$
(2) *If* $Y \in \mathscr{T} \cap \mathscr{U}$, *then* $Y \cap \mathscr{T} \equiv Y \cap \mathscr{U}$.

PROOF. (1) Let $\phi \neq Y \cap O$ where $O \in \mathscr{T}$. By (ii) of theorem 2.1, there exists a $U \in \mathscr{U}$ such that $\phi \neq U \subset O$. Since Y is dense, $\phi \neq Y \cap U \subset Y \cap O$.

(2) Let $\phi \neq Y \cap O$ where $O \in \mathscr{T}$. Then $Y \cap O \in \mathscr{T}$ and hence there exists a $U \in \mathscr{Y}$ such that $\phi \neq U \subset Y \cap O$. Thus $\phi \neq U \cap Y \subset O \cap Y$.

THEOREM 3.2. *Let $(X, \mathscr{T}, \mathscr{U})$ be a bitopological space and $Y \subset X$, $Y \cap \mathscr{T} \equiv Y \cap \mathscr{U}$, $Y \in \mathscr{T} \cap \mathscr{U}$ and Y both $\mathscr{T}$-dense and $\mathscr{U}$-dense. Then $\mathscr{T} \equiv \mathscr{U}$.*

PROOF. Let $\phi \neq O \in \mathscr{T}$; then $\phi \neq Y \cap O$ since Y is dense. Since $Y \cap \mathscr{U} \equiv Y \cap \mathscr{T}$, there exists a $U \in \mathscr{Y}$ such that $\phi \neq Y \cap U \subset Y \cap O$. But $Y \cap U \in \mathscr{U}$ since $Y \in \mathscr{U}$. Thus $\phi \neq Y \cap U \subset O$.

THEOREM 3.3. *Let $(X, \mathscr{T}, \mathscr{U})$ be a bitopological space and $X = \bigcup \{A_\alpha : \alpha \in \Delta\}$ where $A_\alpha \in \mathscr{T} \cap \mathscr{U}$ for each $\alpha \in \Delta$. Then $\mathscr{T} \equiv \mathscr{U}$ iff $A_\alpha \cap \mathscr{T} \equiv A_\alpha \cap \mathscr{U}$ for each $\alpha \in \Delta$.*

PROOF. The necessity follows from (2) of lemma 3.1. To show the sufficiency, let $\phi \neq O \in \mathscr{T}$. Then $O \cap A_\alpha \neq \phi$ for some $\alpha \in \Delta$ and hence there exists a $U \in \mathscr{U}$ such that $\phi \neq A_\alpha \cap U \subset A_\alpha \cap O$. But $A_\alpha \cap U \in \mathscr{U}$ and $\phi \neq A_\alpha \cap U \subset O$. It follows from (ii) of theorem 2.1 that $\mathscr{T} = \mathscr{U}$.

We now obtain the easy

COROLLARY 3.4. *Let $(X, \mathscr{T})$ be the disjoint union of the family of spaces $\{(X_\alpha, \mathscr{T}_\alpha): \alpha \in \Delta\}$ and $(X, \mathscr{U})$ the disjoint union of the fimily $\{(X_\alpha, \mathscr{U}_\alpha): \alpha \in \Delta\}$. Then $\mathscr{T} \equiv \mathscr{U}$ iff $\mathscr{T}_\alpha \equiv \mathscr{U}_\alpha$ for each $\alpha \in \Delta$.*

PROOF. Apply theorem 3.3 using the fact that $X_\alpha \in \mathscr{T} \cap \mathscr{U}$ for each $\alpha \in \Delta$ and $\mathscr{T}_\alpha = X_\alpha \cap \mathscr{T}$, $\mathscr{U}_\alpha = X_\alpha \cap \mathscr{U}$.

COROLLARY 3.5. *Let $(X, \mathscr{T}, \mathscr{U})$ be a bitopological space and $(X^*, \mathscr{T}^*)$, $(X^*, \mathscr{U}^*)$ the one-point compactifications of $(X, \mathscr{T})$ and $(X, \mathscr{U})$ respectively. Then $\mathscr{T}^* \equiv \mathscr{U}^*$ iff $\mathscr{T} \equiv \mathscr{U}$ and $\mathscr{T}$ and $\mathscr{U}$ are both compact or both noncompact.*

PROOF. Suppose $\mathscr{T}^* \equiv \mathscr{U}^*$. Then $\mathscr{T} = X \cap \mathscr{T}^* \equiv X \cap \mathscr{U}^* = \mathscr{U}$ by (2) of lemma 3.1 since $X \in \mathscr{T}^* \cap \mathscr{U}^*$. Thus $\mathscr{T} \equiv \mathscr{U}$. $\mathscr{T}$ is compact iff $\{\infty\} \in \mathscr{T}^*$ iff $\{\infty\} \in \mathscr{U}^*$ iff $\mathscr{U}$ is compact.

Conversely, suppose $\mathscr{T} \equiv \mathscr{U}$ and $\mathscr{T}$ and $\mathscr{U}$ are both noncompact. Then X is $\mathscr{T}^*$-and $\mathscr{U}^*$-dense in X^* and $X \in \mathscr{T}^* \cap \mathscr{U}^*$. By theorem 3.2, $\mathscr{T}^* \equiv \mathscr{U}^*$. Now suppose that $\mathscr{T}$ and $\mathscr{U}$ are both compact. Then $X^* = X \cup \{\infty\}$, $X \in \mathscr{T}^* \cap \mathscr{U}^*$ and $\{\infty\} \in \mathscr{T}^* \cap \mathscr{U}^*$. By theorem 3.3, $\mathscr{T}^* \equiv \mathscr{U}^*$.

LEMMA 3.6. *Let $f:(X, \mathscr{T}) \to (Y, \mathscr{U})$ and $f:(X, \mathscr{T}^*) \to (Y, \mathscr{U}^*)$ be continuous* *open surjections. If $\mathscr{T} \equiv \mathscr{T}^*$, then $\mathscr{U} \equiv \mathscr{U}^*$.*

PROOF. Let $\phi \neq U \in \mathscr{U}$; then $\phi \neq f^{-1}[U] \in \mathscr{T}$ and hence by (ii) of theorem 2.1, there exists an $O^* \in \mathscr{T}^*$ such that $\phi \neq O^* \subset f^{-1}[U]$. Then $\phi \neq f[O^*] \subset U$ and $f[O^*] \in \mathscr{U}^*$.

THEOREM 3.7. *Let $(X_\alpha, \mathscr{T}_\alpha, \mathscr{U}_\alpha)$ be a bitopological space for each $\alpha \in \Delta$ and let $(X, \mathscr{T}) = \times\{(X_\alpha, \mathscr{T}_\alpha): \alpha \in \Delta\}$, $(X, \mathscr{U}) = \times\{(X_\alpha, \mathscr{U}_\alpha): \alpha \in \Delta\}$. Then $\mathscr{T} \equiv \mathscr{U}$ iff $\mathscr{T}_\alpha \equiv \mathscr{U}_\alpha$ for each $\alpha \in \Delta$.*

PROOF. The necessity follows from lemma 3.6. To show the sufficiency, let $\phi \neq \bigcap\{P_{\alpha_i}^{-1}[O_{\alpha_i}]: i=1, \cdots, n\}$ where $O_{\alpha_i} \in \mathscr{T}_{\alpha_i}$. By (ii) of theorem 2.1, there exist $U_{\alpha_i} \in \mathscr{U}_{\alpha_i}$ such that $\phi \neq U_{\alpha_i} \subset O_{\alpha_i}$ Then $\phi \neq \bigcap\{P_{\alpha_i}^{-1}[U_{\alpha_i}]: i=1, \cdots, n\} \subset \bigcap\{P_{\alpha_i}^{-1}[O_{\alpha_i}]: i=1 \cdots, n\}$. Applying (ii) of theorem 2.1, it follows that $\mathscr{T} \equiv \mathscr{U}$.

4. Maximal topologies in $[\mathscr{T}]$

LEMMA 4.1. *Let $(X, \mathscr{T})$ be a topological space. There exists a $\mathscr{U}$ maximal in $[\mathscr{T}]$ such that $\mathscr{T} \subset \mathscr{U}$.*

PROOF. Let $\mathscr{A} = [\mathscr{T}' : \mathscr{T}' \in [\mathscr{T}]$ and $\mathscr{T} \subset \mathscr{T}'\}$; it suffices to show that $\mathscr{A}$ has a maximal element. We apply Zorn's lemma; let $\phi \neq \mathscr{B} \subset \mathscr{A}$, $\mathscr{B}$ a chain. Let $\mathscr{T}^* = \vee\{\mathscr{T}' : \mathscr{T}' \in \mathscr{B}\}$. Clearly $\mathscr{T} \subset \mathscr{T}^*$; it suffices to show that $\mathscr{T}^* \in [\mathscr{T}]$ and hence $\mathscr{T}^* \in \mathscr{A}$. Let $\phi \neq O^* \in \mathscr{T}^*$. There exists an $O' \in \mathscr{T}' \in \mathscr{B}$ such that $\phi \neq O' \subset O^*$. Since $\mathscr{T}' \equiv \mathscr{T}$, there exists an $O \in \mathscr{T}$ such that $\phi \neq O \subset O' \subset O^*$ and hence $\phi \neq O \subset O^*$. By (ii) of theorem 2.1, $\mathscr{T} \equiv \mathscr{T}^*$.

LEMMA 4.2. *Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. Then $\mathscr{T} \equiv \mathscr{T}[A]$ iff $A \in \varsigma(\mathscr{T})$ (see definition 1.4 and 5], page 93).*

PROOF. Sufficiency. We employ (ii) of theorem 2.1. Let $\phi \neq W \in \mathscr{T}[A]$; then there exist $O, U \in \mathscr{T}$ such that $W = O \cup (U \cap A)$ (see [2]). If $\phi \neq O$, then $\phi \neq O \subset W$ and there is nothing more to prove. If $\phi = O$, then $U \cap A \neq \phi$; let $x \in U \cap A$. Since $A \subset c_t \operatorname{Int}_t A$, it follows that $U \cap \operatorname{Int}_t A \neq \phi$. Take $O^* = U \cap \operatorname{Int}_t A$. Then $O^* \in \mathscr{T}$ and $\phi \neq O^* \subset U \cap A$.

Necessity. Suppose $A \notin \varsigma(\mathscr{T})$; then $A \not\subset c_t \operatorname{Int}_t A$. Let $a \in A$, $a \notin c_t \operatorname{Int}_t A$. There exists then an $O \in \mathscr{T}$ such that $a \in O$ and $O \cap \operatorname{Int}_t A = \phi$. But $\phi \neq O \cap A \in \mathscr{T}[A]$ and hence there exists an $O^* \in \mathscr{T}$ such that $\phi \neq O^* \subset O \cap A$. Hence $O \cap \operatorname{Int}_t A \supset O \cap O^* \supset O^* \neq \phi$ and $O \cap \operatorname{Int}_t A \neq \phi$, a contradiction.

LEMMA 4.3. *Let $(X, \mathscr{T})$ be a toplogical space and $A \subset X$. Then $\mathscr{T} \neq \mathscr{T}[A]$ iff $A \notin \mathscr{T}$.*

We omit the easy proof.

THEOREM 4.4. *Let $(X, \mathscr{T})$ be a topological space. Then $\mathscr{T}$ is maximal in $[\mathscr{T}]$ iff $\mathscr{T} = \varsigma(\mathscr{T})$.*

PROOF. Necessity. Suppose $A \in \varsigma(\mathscr{T})$, but $A \notin \mathscr{T}$. The $\mathscr{T} \equiv \mathscr{T}[A]$ by lemma 4.2 and $\mathscr{T} \neq \mathscr{T}[A]$ by lemma 4.3. But $\mathscr{T} \subset \mathscr{T}[A]$ and hence $\mathscr{T}$ is not maximal in $[\mathscr{T}]$.

Sufficiency. Suppose $\mathscr{T}$ is not maximal in $[\mathscr{T}]$; there exists then a $\mathscr{U} \in [\mathscr{T}]$ such that $\mathscr{T} \subset \mathscr{U}$, $\mathscr{T} \neq \mathscr{U}$. Take $U \in \mathscr{U} - \mathscr{T}$. Then $\mathscr{T} \subset \mathscr{T}[U] \subset \mathscr{U}$ and $\mathscr{T} \neq \mathscr{T}[U]$ by lemma 4.3. By lemma 2.5, $\mathscr{T} \equiv \mathscr{T}[U]$ and by lemma 4.2, $U \in \varsigma(\mathscr{T}) = \mathscr{T}$ and $U \in \mathscr{T}$, a contradiction.

COROLLARY 4.5. *If $\mathscr{T}$ is maximal in $[\mathscr{T}]$, then $\mathscr{T}$ is an extremally disconnected topology.*

PROOF. Let $O \in \mathscr{T}$; then $cO \in \varsigma(\mathscr{T}) = \mathscr{T}$ by theorem 4.4.

COROLLARY 4.6. *Let $(X, \mathscr{T})$ be a topological space. There exists a topology $\mathscr{U}$ on X such that $\mathscr{T} \subset \mathscr{U}$, $\mathscr{T} \equiv \mathscr{U}$ and $\mathscr{U}$ is extremally disconnected.*

The proof follows from lemma 4.1 and corollary 4.5.

COROLLARY 4.7. *Let $\mathscr{T}$ be a non D-topology on X (see definition* 1.1*) There exists a topology $\mathscr{U}$ on X such that $\mathscr{T} \subset \mathscr{U}$, $\mathscr{U} \equiv \mathscr{T}$ and $\mathscr{U}$ is disconnected.*

PROOF. By lemma 4.1, there exists a topology $\mathscr{U}$ on X such that $\mathscr{T} \subset \mathscr{U}$, $\mathscr{T} \equiv \mathscr{U}$ and $\mathscr{U}$ is maximal in $[\mathscr{T}]$. We now show that $\mathscr{U}$ is disconnected. Since $\mathscr{T}$ is a non D-topology, there exists nonempty disjoint open set O_1 and O_2. Thus O_1 and $\mathrm{Int}_t \mathscr{C} O_1$ are nonempty and $O_1 \cup \mathrm{Int}_t \mathscr{C} O_1$ is $\mathscr{T}$-dense. By corollary 2.2 there exist U_1 and U_2 in $\mathscr{U}$ such that $U_1 \subset O_1$, $U_2 \subset \mathrm{Int}_t \mathscr{C} O_1$, $c_t U_1 = c_t O_1$ and $c_t U_2 = c_t \mathrm{Int}_t \mathscr{C} O_1$. Thus $U_1 \cup U_2$ is $\mathscr{T}$-dense and hence $\mathscr{U}$-dense since $\mathscr{T} \equiv \mathscr{U}$. Hence $X = c_u U_1 \cup c_u U_2$ and since $\mathscr{U}$ is extremally disconnected by corollary 4.5, it follows that $c_u U_1 \cap c_u U_2 = \phi$. Thus $\mathscr{U}$ is a disconnected topology.

COROLLARY 4.8. *Let $(X, \mathscr{T})$ be a topological space. Then $\mathscr{T}$ is a D-topology iff $[\mathscr{T}]$ consists only of connected topologies.*

The proof follows from corollary 4.6 and (iii) of theorem 2.3.

5. When is there a maximum in $[\mathcal{T}]$?

THEOREM 5.1. *Let $\mathcal{T}$ be an S-topology on X (see definition 1.2). Then $\mathcal{T}$ is the largest topology in $[\mathcal{T}]$.*

PROOF. Let $\mathcal{U}\in[\mathcal{T}]$; we show that $\mathcal{U}\subset\mathcal{T}$. Let $\phi\neq U\in\mathcal{U}$; by (ii) of theorem 2.1, there exists an $O\in\mathcal{T}$ such that $\phi\neq O\subset U$. Since $\mathcal{T}$ is an S-topology, $U\in\mathcal{T}$.

THEOREM 5.2. *Let $(X, \mathcal{T})$ be a topological space. Then $[\mathcal{T}]$ has a maximum iff $\mathcal{T}_1$, $\mathcal{T}_2$ in $[\mathcal{T}]$ implies that $\mathcal{T}_1\vee\mathcal{T}_2\in[\mathcal{T}]$.*

PROOF. Necessity. Suppose that $\mathcal{U}$ is the largest member of $[\mathcal{T}]$ and let $\mathcal{T}_1$, $\mathcal{T}_2\in[\mathcal{T}]$. Then $\mathcal{T}_1\subset\mathcal{T}_1\vee\mathcal{T}_2\subset\mathcal{U}$ and by lemma 2.5, $\mathcal{T}_1\vee\mathcal{T}_2\equiv\mathcal{U}\equiv\mathcal{T}$.

Sufficiency. By lemma 4.1, there exists a topology $\mathcal{U}$ such that $\mathcal{T}\subset\mathcal{U}$ and $\mathcal{U}$ is maximal in $[\mathcal{T}]$. We show that $\mathcal{U}$ is the maximum in $[\mathcal{T}]$. Let $\mathcal{T}^*\equiv\mathcal{T}$. Then $\mathcal{T}^*\vee\mathcal{U}\equiv\mathcal{U}$ and $\mathcal{U}\subset\mathcal{T}^*\vee\mathcal{U}$. Since $\mathcal{U}$ is maximal in $[\mathcal{T}]$, it follows that $\mathcal{T}^*\subset\mathcal{T}^*\vee\mathcal{U}=\mathcal{U}$.

COROLLARY 5.3. *Let $\mathcal{T}$ be a D-topology on X. Then $[\mathcal{T}]$ has a maximum.*

PROOF. Let $\mathcal{T}_1,\mathcal{T}_2\in[\mathcal{T}]$; we show that $\mathcal{T}_1\vee\mathcal{T}_2\in[\mathcal{T}]$. Firstly, let $\phi\neq O\in\mathcal{T}$. By (ii) of theorem 2.1, there exists an $O_1\in\mathcal{T}_1$ such that $\phi\neq O_1\subset O$. Then $O_1\in\mathcal{T}_1\subset\mathcal{T}_1\vee\mathcal{T}_2$. Conversely, let $\phi\neq O_1\cap O_2\in\mathcal{T}_1\vee\mathcal{T}_2$ where $O_i\in\mathcal{T}_i$. There exist then $O_i^*\in\mathcal{T}$ such that $\phi\neq O_i^*\subset O_i$. But $\phi\neq O_1^*\cap O_2^*$ since $\mathcal{T}$ is a D-topology and $O_1^*\cap O_2^*\subset O_1\cap O_2$.

THEOREM 5.4. *Let $\mathcal{T}$ be a D-topology and let $\mathcal{U}$ be the maximum in $[\mathcal{T}]$. Then $\mathcal{U}$ is an S topology.*

PROOF. Let $\phi\neq U\subset A$ where $U\in\mathcal{U}$, We must show that $A\in\mathcal{U}$. Suppose on the contrary that $A\notin\mathcal{U}$. Since $\mathcal{U}$ is maximal, $A\not\subset c_u\ \mathrm{Int}_u\ A$ by theorem 4.4. Thus, $c_u\ \mathrm{Int}_u\ A\neq X$ and $\mathrm{Int}_u\ A$ is not $\mathcal{U}$-dense. But $\mathcal{U}$ is a D-topology by (iii) of theorem 2.3 and $\phi\neq U\subset\mathrm{Int}_u\ A$. Then $\mathrm{Int}_u\ A$ is $\mathcal{U}$-dense, a contradiction.

6. Minimal and minimum topologies in $[\mathcal{T}]$

THEOREM 6.1 *Let $(X, \mathcal{T})$ be a topological space. Then $[\mathcal{T}]$ has a minimum element iff for every nonempty family $\{\mathcal{T}_\alpha : \alpha\in\Delta\}$ in $[\mathcal{T}]$, then $\bigcap\{\mathcal{T}_\alpha : \alpha\in\Delta\}\in[\mathcal{T}]$.*

PROOF. Sufficiency. $\bigcap\{\mathcal{T}' : \mathcal{T}'\in[\mathcal{T}]\}$ is the smallest member of $[\mathcal{T}]$.

Necessity. Let $\mathscr{U}$ be the minimum element of $[\mathscr{T}]$ and suppose that $\{\mathscr{T}_\alpha : \alpha\in\Delta\}$ is a nonempty family in $[\mathscr{T}]$. Then $\mathscr{U}\subset\bigcap\{\mathscr{T}_\alpha : \alpha\in\Delta\}\subset\mathscr{T}_{\alpha^*}$ where $\alpha^*\in\Delta$. By lemma 2.5, $\bigcap\{\mathscr{T}_\alpha : \alpha\in\Delta\}\equiv\mathscr{U}\in[\mathscr{T}]$.

THEOREM 6.2 *Let $(X, \mathscr{T})$ be a T_1, nondiscrete topological space. Then $\mathscr{T}$ is not minimal in $[\mathscr{T}]$.*

PROOF. Let $\{x^*\}\notin\mathscr{T}$ and define $\mathscr{U}=\{O : O\in\mathscr{T}, O=X$ or $x^*\notin O\}$. Then $\mathscr{U}$ is a topology and $\mathscr{U}\subset\mathscr{T}$. Let $x\neq x^*$; then $\mathscr{C}\{x\}\in\mathscr{T}$, but $\mathscr{C}\{x\}\notin\mathscr{U}$ and hence $\mathscr{U}\neq\mathscr{T}$. We employ (ii) of theorem 2.1 to show that $\mathscr{T}\equiv\mathscr{U}$; let $\phi\neq O\in\mathscr{T}$. If $x^*\notin O$, then $O\in\mathscr{U}$; if $x^*\in O$, then $\phi\neq O\cap\mathscr{C}\{x^*\}\subset O$ and $O\cap\mathscr{C}\{x^*\}\in\mathscr{U}$.

7. A sufficient condition for $[\mathscr{T}]=\{\mathscr{T}\}$

THEOREM 7.1. *If $\mathscr{T}$ has a basis of minimal open sets, then $[\mathscr{T}]=\{\mathscr{T}\}$.*

PROOF. Let $\mathscr{T}\equiv\mathscr{U}$. We first show that $\mathscr{T}\subset\mathscr{U}$. Let $x\in O\in\mathscr{T}$; there exists then a minimal open set $O^*\in\mathscr{T}$ such that $x\in O^*\subset O$. By (ii) of theorem 2.1, there exists a $U^*\in\mathscr{U}$ such that $\phi\neq U^*\subset O^*$ and there exists an $O^\#\in\mathscr{T}$ such that $\phi\neq O^\#\subset U^*\subset O^*$. Since O^* is minimal, $O^\#=O^*$ and hence $x\in U^*\subset O$. Thus $O\in\mathscr{U}$.

Next we show that $\mathscr{U}\subset\mathscr{T}$. Let $x\in U\in\mathscr{U}$; let $x\in O$ where O is a minimal $\mathscr{T}$-open set. Using the above argument, it follows that O is a minimal $\mathscr{U}$-open set and hence $x\in O\subset U$. Thus $U\in\mathscr{T}$.

COROLLARY 7.2 *Let $(X, \mathscr{T})$ be a topological space in which the open sets and the closed sets coincide. Then $[\mathscr{T}]=\{\mathscr{T}\}$.*

PROOF. $\{c(x) : x\in X\}$ is a basis for $\mathscr{T}$ consisting of minimal open sets.

8. Some examples

EXAMPLE 8.1 Let X be the reals, $\mathscr{T}$ the usual topology, $\mathscr{U}$ the topology having sets of the form $[a, b)$ as base and $\mathfrak{V}$ the topology having sets of the form $(a, b]$ as base. Then $\mathscr{T}\equiv\mathscr{U}\equiv\mathfrak{V}$ follows from (ii) of theorem 2.1. $[\mathscr{T}]$ has no maximum since $\mathscr{U}\vee\mathfrak{V}=\mathscr{P}(X)\neq\mathscr{T}$ (see theorem 5.2). $(X, \mathscr{U}$ is 0-dimensional, $(X, \mathscr{T})$ is not. $\mathscr{T}$ is locally connected and connected whereas $\mathscr{U}$ is totally disconnected. $\mathscr{T}$ is locally compact, but $\mathscr{U}$ is not. $\mathscr{T}$ is a second axiom space and $\mathscr{U}$ is not. $\mathscr{T}$ is metric, $\mathscr{U}$ is not metrizable.

EXAMPLE 8.2. Let $X=\{a, b\}$ and $\mathscr{T}=\{\phi, \{a\}, X\}$. Then $[\mathscr{T}]=\{\mathscr{T}\}$, but $\mathscr{T}$ does not have a basis of minimal open sets (see theorem 7.1).

EXAMPLE 8.3. Let X be the reals and $\mathcal{T}=\{O: O\neq\phi$ or $O=X$ or $O=(\infty, a)$ for some $a\in X\}$. Then $[\mathcal{T}]$ has no minimum element; let $\mathcal{U}=\{U: U=\phi$ or $U=X$ or $U=(-\infty, -2n)$ $n=1, 2, \cdots\}$ and $\mathcal{V}=\{V: V=\phi$ or $V=X$ or $V=(-\infty, -(2n+1))$ $n=1, 2, \cdots\}$. Then $\mathcal{U}\equiv\mathcal{T}\equiv\mathcal{V}$, but $\mathcal{U}\cap\mathcal{V}=\{\phi, X\}\not\equiv\mathcal{T}$ (see theorem 6.1).

EXAMPLE 8.4 Let $X=\{a, b, c\}$ and $\mathcal{T}=\{\phi, \{a\}, X\}$, $\mathcal{U}=\{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then $\mathcal{T}\equiv\mathcal{U}$, $\mathcal{T}$ is normal, $\mathcal{U}$ is not normal. If $Y=\{b, c\}$, then $Y\cap\mathcal{T}\not\equiv Y\cap\mathcal{U}$.

EXAMPLE 8.5 Let X be the positive integers, $\mathcal{T}=\{O: 1\notin O$ or $1\in O$ and $\mathscr{C}O$ is finite$\}$ and $\mathcal{U}=\{U: 1\notin U$ or $U=X\}$. Then $\mathcal{T}\equiv\mathcal{U}$, $\mathcal{T}$ is compact Hausdorff and hence completely regular, $\mathcal{U}$ is not T_1($\{2\}$ is not closed) nor is it regular ($2\notin\mathscr{C}\{2\}$ and $\mathscr{C}\{2\}$ is closed, but 2 and $\mathscr{C}\{2\}$ cannot be separated by $\mathcal{U}$-open sets).

EXAMPLE 8.6. Let $(X, \mathcal{T})$ be as in example 8.5 and let $\mathcal{U}$ be the topology for X generated by $\{1, 2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\cdots$ as base. Then $\mathcal{T}\equiv\mathcal{U}$, $\mathcal{T}$ is compact Hausdorff, $\mathcal{U}$ is not compact nor is it a T_0-space.

EXAMPLE 8.7. Let $X=\{a, b, c\}$ and $\mathcal{T}=\{\phi, \{a\}, \{b, c\}, X\}$. Then $[\mathcal{T}]=\{\mathcal{T}\}$ by corollary 7.2. Note that $\mathcal{T}$ is not an S-topology and hence the converse of theorem 5.1 is false.

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