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ON THE DISTORTION THEOREMS I

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0. Abstract

The coefficient problems of univalent functions was given by Bieberbach. As is well-known, Koebe distortion theorem has close connection with the coefficient problems of univalent functions. It is purpose of this paper to give the distortion theorems for fractional integral and derivative of univalent functions.

1. Definitions of the fractional integral and derivative

There are many definitions of the fractional integral and derivative. At first, Liouville [1] defined the fractional integral of order α by

$$D^{-\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt,$$

where $f(x)$ is continuous in $[a, b]$ and $\alpha>0$.

Recently, Osler [2], [3] defined the fractional derivative of order α by

$$D_z^\alpha f(z)=\frac{\Gamma(\alpha+1)}{2\pi i}\int_0^{(z^+)} \frac{f(\zeta)}{(\zeta-z)^{\alpha+1}}d\zeta,$$

where $f(z)=z^p g(z)$, where $g(z)$ is analytic in a simply connected region of the z-plane containing the origin and $\mathrm{Re}(p)>-1$, the multiplicity of $(\zeta-z)^{-\alpha-1}$ is removed by requiring $\ln(\zeta-z)$ to be real when $(\zeta-z)>0$, and symbol (z^+) means the integral path is enclosing z once in the positive sense.

Moreover, Nishimoto [4] defined the fractional derivative of order α by

$$_{C^-}f_\alpha(z)=\frac{\Gamma(\alpha+1)}{2\pi i}\int_{C^-} \frac{f(\zeta)}{(\zeta-z)^{\alpha+1}}d\zeta,$$

$$_{C^+}f_\alpha(z)=\frac{\Gamma(\alpha+1)}{2\pi i}\int_{C^+} \frac{f(\zeta)}{(\zeta-z)^{\alpha+1}}d\zeta,$$

$$_C f_{-n}(z)=\lim_{\alpha\to -n} {}_C f_\alpha(z),$$

where $f(z)$ is an analytic function that has no branch point inside of C, C^- and C^+, which are the integral curves which are shown in Fig. 1 and Fig. 2, respectively, and n is any positive integer.

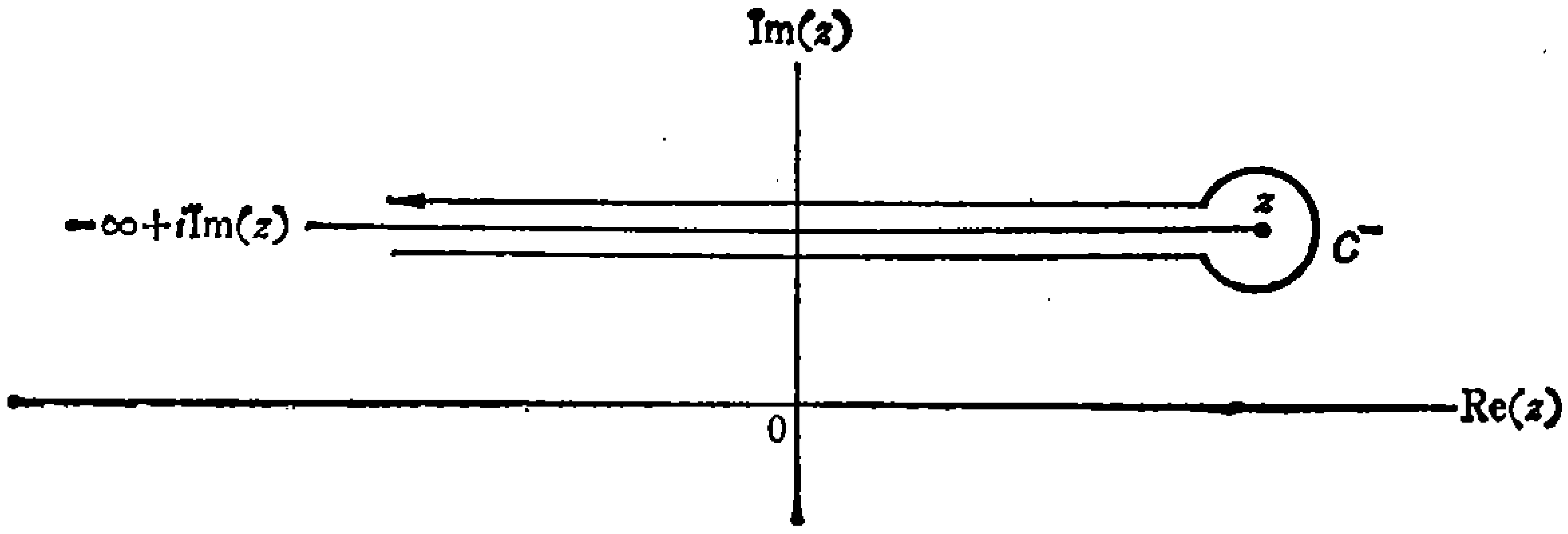

Fig. 1

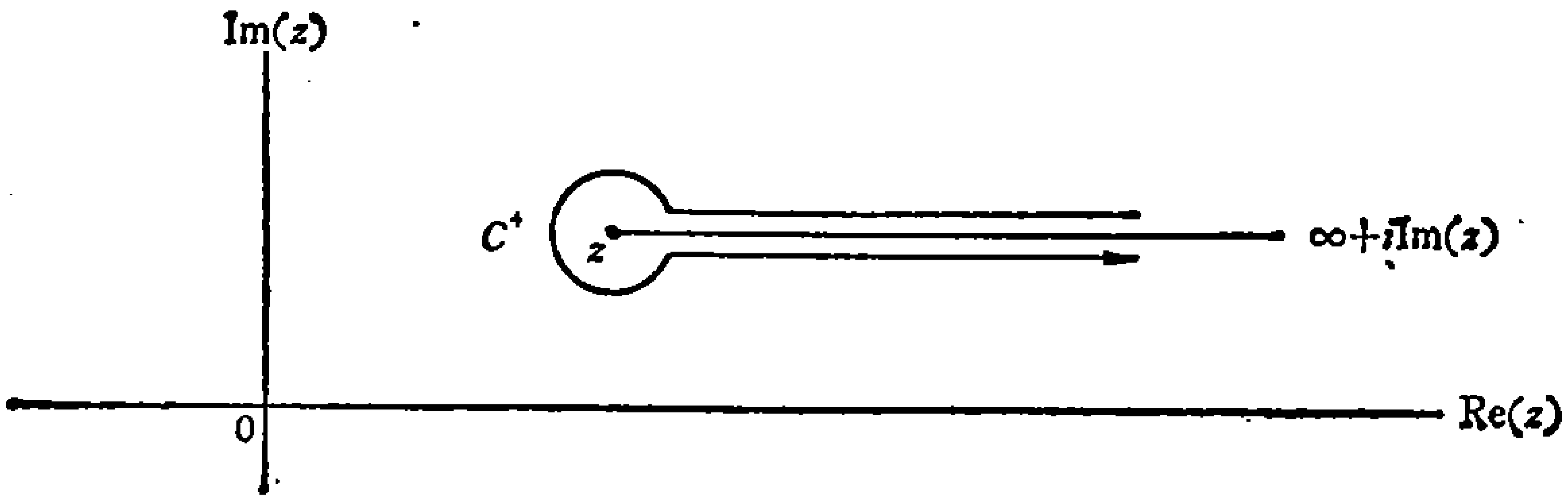

Fig. 2

In this paper, the fractional integral and derivative of order α are defined by the following.

DEFINITION 1. The *fractional integral of order α* is defined by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta)}{(\zeta-z)^{1-\alpha}} d\zeta,$$

$$f(z) = \lim_{\alpha \to 0} D_z^{-\alpha} f(z),$$

where α is a positive real number, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(\zeta-z)^{\alpha-1}$ is removed by requiring $\ln(\zeta-z)$ to be real when $(\zeta-z)>0$.

DEFINITION 2. The *fractional derivative of order α is* defined by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(\zeta-z)^{\alpha}} d\zeta,$$

$$f(z) = \lim_{\alpha \to 0} D_z^{\alpha} f(z),$$

$$f'(z)=\lim_{\alpha\to 1} D_z^{\alpha} f(z),$$

where $0<\alpha<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(\zeta-z)^{-\alpha}$ is removed by requiring $\ln(\zeta-z)$ to be real when $(\zeta-z)>0$.

2. The class of univalent functions

Let S denote the class of functions

(1) $$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

are analytic and univalent in the unit disk, S^* denote the class of functions (1) are univalent starlike with respect to the origin in the unit disk, and C denote the class of functions (1) are univalent convex in the unit disk.

A necessary and sufficient condition for $f(z)\in S$ to be univalent starlike in the unit disk is

(2) $$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0, \quad |z|<1.$$

A necessary and sufficient condition for $f(z)\in S$ to be univalent convex in the unit disk is

(3) $$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0, \quad |z|<1.$$

For the functions $f(z)$ of the form (1), the simple computations give

(4) $$F(z)=\Gamma(2+\alpha)z^{-\alpha}D_z^{-\alpha}f(z)$$

$$=z+\sum_{n=2}^{\infty}\frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)}a_n z^n,$$

where α is a positive real number, and

(5) $$G(z)=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha}f(z)$$

$$=z+\sum_{n=2}^{\infty}\frac{n!\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_n z^n,$$

where $0<\alpha<1$.

THEOREM 1. *There are the univalent starlike functions of the form* (1) *in the unit disk such that* $F(z)\in S^*$

EXAMPLE 1. Let

(6) $$f(z)=z+a_2 z^2\in S^*.$$

Then, from (2), it is clear that $2|a_2|<1$.

Therefore,

$$\mathrm{Re}\left\{\frac{zF'(z)}{F(z)}\right\}=\frac{(2+\alpha)^2+6(2+\alpha)|a_2||z|\cos(\theta+\phi)+8|a_2|^2|z|^2}{(2+\alpha)^2+4(2+\alpha)|a_2||z|\cos(\theta+\phi)+4|a_2|^2|z|^2}$$
$$\geqq\frac{(2+\alpha-2|a_2||z|)(2+\alpha-4|a_2||z|)}{(2+\alpha+2|a_2||z|)^2}$$
$$\geqq 0$$

for $|z|<1$, where $z=|z|e^{i\theta}$ and $a_2=|a_2|e^{i\phi}$.

REMARK 1. If the function (6) belongs to S^*, then $F(z)$ is an univalent convex function in $|z|<(2+\alpha)/4$, where α is a positive real number.

THEOREM 2. *There are the univalent convex functions of the form* (1) *in the unit disk such that* $F(z)\in C$.

EXAMPLE 2. Let the function (6) belong to C. Then, the necessary and sufficient condition (3) gives $4|a_2|<1$.

Therefore,

$$\mathrm{Re}\left\{1+\frac{zF''(z)}{F'(z)}\right\}=\frac{(2+\alpha)^2+12(2+\alpha)|a_2||z|\cos(\theta+\phi)+32|a_2|^2|z|^2}{(2+\alpha)^2+8(2+\alpha)|a_2||z|\cos(\theta+\phi)+16|a_2|^2|z|^2}$$
$$\geqq\frac{(2+\alpha-4|a_2||z|)(2+\alpha-8|a_2||z|)}{(2+\alpha+4|a_2|\,|z|)^2}$$
$$>0$$

for $|z|<1$, where $z=|z|e^{i\theta}$ and $a_2=|a_2|e^{i\phi}$.

THEOREM 3. *There are the univalent convex functions of the form* (1) *in the unit disk such that* $G(z)\in S^*$.

EXAMPLE 3. Let the function (6) belong to C. Then,

$$\mathrm{Re}\left\{\frac{zG'(z)}{G(z)}\right\}\geqq\frac{(2-\alpha-2|a_2||z|)(2-\alpha-4|a_2||z|)}{(2-\alpha+2|a_2||z|)^2}$$
$$>0$$

for $|z|<1$, where $z=|z|e^{i\theta}$ and $a_2=|a_2|e^{i\phi}$.

REMARK 2. If the function (6) belongs to C, then $G(z)$ is an univalent convex function in $|z|<(2-\alpha)/2$, where $0<\alpha<1$.

REMARK 3. If the function (6) belongs to S^*, then $G(z)$ is an univaient starlike function in $|z|<(2-\alpha)/2$, where $0<\alpha<1$.

REMARK 4. If the function (6) belongs to S^*, then $G(z)$ is an univalent convex function in $|z|<(2-\alpha)/4$, where $0<\alpha<1$.

Now, let S_F^* denote the class of functions of the form (1) that $F(z)\in S^*$, C_F denote the class of functions of the form (1) that $F(z)\in C$ and S^*_G denote the class of functions of the form (1) that $G(z)\in S^*$.

3. The distortion theorems for analytic univalent functions

Owa [5] gave the folllowing conjecture for fractional derivative of univalent functions.

CONJECTURE. Let $f(z)\in S$. Then, perhaps, for any nonnegative α and $|z|<1$,

$$|D_z^\alpha f(z)|\leqq\frac{\Gamma(\alpha+1)(\alpha+|z|)}{(1-|z|)^{\alpha+2}}.$$

Now, the following theorems are given by Theorem 1, Theorem 2, and Theorem 3.

THEOREM 4. *If $f(z)\in S_F^*$, then for $|z|<1$,*

$$\frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)(1+|z|)^2}\leqq|D_z^{-\alpha}f(z)|\leqq\frac{|z|^{1+\alpha}}{\Gamma(2+\alpha)(1-|z|)^2},$$

where α is a positive real number. Equality holds for Koebe function

$$f(z)=\frac{z}{(1-e^{i\theta}z)^2}.$$

PROOF. Since the function $F(z)$ belongs to S^*,

$$\frac{|z|}{(1+|z|)^2}\leqq|\Gamma(2+\alpha)z^{-\alpha}D_z^{-\alpha}f(z)|\leqq\frac{|z|}{(1-|z|)^2}.$$

Hence, Theorem 4 is right.

COROLLARY 1. *If $f(z)\in S_F^*$, then $D_z^{-\alpha}f(z)$ contains the disk with center at the origin and radius* $1/4\Gamma(2+\alpha)$.

THEOREM 5. *If $f(z)\in C_F$, then for $|z|<1$,*

$$\frac{|z|^\alpha}{\Gamma(2+\alpha)(1+|z|)}\leqq|D_z^{-\alpha}f(z)|\leqq\frac{|z|^\alpha}{\Gamma(2+\alpha)(1-|z|)},$$

where α is a positive real number. Equality holds for the function

$$f(z)=\frac{z}{1-z}.$$

COROLLARY 2. *If $f(z)\in C_F$, then $D_z^{-\alpha}f(z)$ contains the disk with center at the origin and radius $1/2\Gamma(2+\alpha)$.*

THEOREM 6. *If $f(z)\in S_G^*$, then for $|z|<1$,*

$$\frac{|z|^{-\alpha}}{\Gamma(2-\alpha)(1+|z|)}\leqq|D_z^\alpha f(z)|\leqq\frac{|z|^{-\alpha}}{\Gamma(2-\alpha)(1-|z|)},$$

where $0<\alpha<1$. Equality holds for the function

$$f(z)=\frac{z}{1-z}.$$

COROLLARY 3. *If $f(z)\in S_G^*$, then $D_z^\alpha f(z)$ contains the disk with center at the origin and radius $1/2\Gamma(2-\alpha)$.*

The proofs of Theorem 5 and Theorem 6 are given much the same way as Theorem 4.

The following theorem holds for the conjecture in [5].

THEOREM 7. *If $f(z)\in S_G^*$, then for $0<\alpha<1$ and*

$$\frac{(\alpha^4-\alpha^3-1)+\sqrt{\alpha^8-2\alpha^7+\alpha^6-2\alpha^4-2\alpha^3+4\alpha^2+1}}{2\alpha^2(1-\alpha)}\leqq|z|<1,$$

$$|D_z^\alpha f(z)|\leqq\frac{\Gamma(\alpha+1)(\alpha+|z|)}{(1-|z|)^{\alpha+2}}.$$

PROOF.

$$\frac{|z|^{-\alpha}}{\Gamma(2-\alpha)(1-|z|)}-\frac{\Gamma(\alpha+1)(\alpha+|z|)}{(1-|z|)^{\alpha+2}}$$

$$=\frac{(1-|z|)^{\alpha+1}-\Gamma(\alpha+1)\Gamma(2-\alpha)|z|^\alpha(\alpha+|z|)}{\Gamma(2-\alpha)|z|^\alpha(1-|z|)^{\alpha+2}}.$$

Let

$$P(\alpha,\ |z|)=(1-|z|)^{\alpha+1}-\Gamma(\alpha+1)\Gamma(2-\alpha)|z|^\alpha(\alpha+|z|).$$

Then,

$$P(\alpha,\ |z|)=(1-|z|)-\alpha(1-\alpha)\frac{\alpha}{\sin\pi\alpha}|z|^\alpha(\alpha+|z|)$$

$$<(1-|z|)-\alpha^2(1-\alpha)|z|(\alpha+|z|).$$

Therefore, $P(\alpha, |z|)$ is negative for $0<\alpha<1$ and

$$\frac{(\alpha^4-\alpha^3-1)+\sqrt{\alpha^8-2\alpha^7+\alpha^6-2\alpha^4-2\alpha^3+4\alpha^2+1}}{2\alpha^2(1-\alpha)} \leqq |z| < 1.$$

Hence, Theorem 7 is established.

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