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WHICH SELFADJOINT OPERATOR IN THE DOMAIN OF A CLOSED DERIVATION SATISFIES THE DOMAIN PROBLEM?

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1. Introduction

A derivation δ in a C^*-algebra $\mathfrak{A}$ is a linear mapping from a dense *-subalgebra $\mathscr{D}(\delta)\subset\mathfrak{A}$ into $\mathfrak{A}$ satisfying the property

$$\delta(AB)=\delta(A)B+A\delta(B),\quad A,B\in\mathscr{D}(\delta).$$

If the domain of δ $\mathscr{D}(\delta)=\mathfrak{A}$, then it is proved by Sakai [1] that δ is norm continuous. However many interesting derivations are unbounded ones. Especially the most important derivation d/dx is an unbounded one. Sakai [2] initiated a study of unbounded derivations in C^*-algebras. Powers [8] stated the following theorem: Let us assume $\mathfrak{A}$ is a C^*-algebra with unit and δ is a closed densely defined derivation in $\mathfrak{A}$, whose domain is denoted by $\mathscr{D}(\delta)$. Suppose $A=A^*$, $A\in\mathscr{D}(\delta)$, and f is a complex valued continuously differentiable function on a closed interval containing the spectrum of A, then $f(A)\in\mathscr{D}(\delta)$.

But Robinson notes that Powers' proof is incomplete. So the following problem, called domain problem, has been remained unsolved: Let A, and f be as above, then is it true that $f(A)\in\mathscr{D}(\delta)$?

In his dissertation [3], the author showed that there is very close relationship between this problem and spectral theory of non self-adjoint operators. Especially he was able to show that the domain problem is true if f is 3rd times continuously differentiable. And Bratteli and Robinson [4] got a little stronger result, saying the theorem is true if f is 2nd times continuously differentiable, modifying Powers argument. Also the author made a conjecture that the theorem would be true. He based this conjecture on one famous problem of spectral theory, called Kantorovitz conjecture. Recently McIntosh [5] found a counter example to the author's conjecture, finally solving the domain problem negatively, and hence also Kantorovitz conjecture is solved negatively. But it is trivial to see that if the C^*-algebra is commutative, then the domain problem holds. Therefore the next question is the following (due to prof. Sakai, private communication): Is the commutativity characteristic of domain problem?

In this paper we give an answer to the question in the title. i.e. We give a necessary and sufficient condition for $A=A^*\in\mathscr{D}(\delta)$ to satisfy that if f is continuously differentiable, then $f(A)\in\mathscr{D}(\delta)$. As a corollary, we show that if A and $\delta(A)$ commutes then the domain problem holds.

2. We know that there are two kinds of spectral theories. Dunford and Schwartz [6] style, and Colojoara and Foiaç [7]. The former is delicate and perfect, but not so practical in cases of bounded operators. The latter is a little rough, but much more practical. In the former, we have necessary and sufficient conditions, but in the latter only sufficient conditions. In his dissertation, we made use of the Colojoara and Foiaç' results. In this paper we are going to use spectral theory in the form developed by Dunford and Schwartz.

Let us start with several definitions and notations in Dunford-Schwartz. $\mathscr{H}$ is a Hilbert space and $x\in\mathscr{H}$. T is an operator on $\mathscr{H}$. We use the symbol $R(\xi;T)$ for $(\xi I-T)^{-1}$ of T at the point ξ in the resolvent set $\rho(T)$. Then an analytic extension of $R(\xi;T)x$ will be an $\mathscr{H}$-valued function f defined and analytic on an open set $D(f)\supset\rho(T)$ and such that $(\xi I-T)f(\xi)=x$, $\xi\in D(f)$. Then $f(\xi)=R(\xi;T)x$ if $\xi\in\rho(T)$.

DEFINITION 2.1. The function $R(\xi;T)x$ is said to *have the single valued extension property* provided that for every pair f, g of analytic extensions of $R(\xi;T)x$ we have $f(\xi)=g(\xi)$ for every ξ in $D(f)\cap D(g)$. The union of the sets $D(f)$ as f varies over all analytic extensions of $R(\xi:T)x$ is called the *resolvent set* of x and is denoted by $\rho_T(x)$. The spectrum $\sigma_T(x)$ of x is defined to be the complement of $\rho(x)$.

REMARK. It is trivial to see that any operator T on a Hilbert space $\mathscr{H}$ having spectrum in a Jordan curve has the single valued extension property.

DEFINITION. 2.2. The operator T is said to *satisfy the fundamental boundedness condition* if there is a constant K, depending only on T, such that for every pair x, y of vectors with $\sigma_T(x)$, $\sigma_T(y)$ disjoint we have $\|x\|\le K\|x+y\|$.

DEFINITION. 2.3. The operator T is said to *satisfy the growth condition* (G_m) if its spectrum lies in a Jordan curve Γ_0 and if, for some constant M, $\|(\xi-\xi_0)^m R(\xi:T)\|\le M$, $\xi\notin\Gamma_0$, $0<|\delta|\le 1$ where ξ_0 is the intersection of the transversal Δ_ξ with the curve Γ_0.

DEFINITION. 2.4. The operator T is *of type* $m-1$ if for any C^{m-1} function f

defined on an open interval containing spec(T) $f(T)$ is defined and moreover if $f_n \to f$ in C^{m-1}-topology, then $f_n(T) \to f(T)$ in operator norm topology.

If δ is a derivation, it is easy to see that

$A \to T = \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix}$ is an algebra homomorphism. (it was observed by Kaplansky).

We can define $f(T)$ if f is analytic (Dunford [6]) and if $f_n \to f$ uniformly where f_n, f analytic, then $f_n(T) \to f(T)$. If we choose f_n to be the polynomial sequence giving Runge's approximation to f, then the closedness of δ says $f(A) \in \mathscr{D}(\delta)$. This is the way we proved this result in the dissertation. Let us try to extend this methodology to C^1-function f. If we could show $T = \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix}$ is of type 1 as defined above, then the closedness of δ gives $f(A) \in \mathscr{D}(\delta)$.

Now let us give a main theorem of spectral theory

THEOREM 2.5. [6, XVI. 5.18] *A bounded linear operator T in Hilbert space whose spectrum lies in the Jordan curve T_0 will be a spectral operator of type $m-1$ if and only if both T and its adjoint satisfy boundedness condition and growth condition (Gm).*

PROOF. See Dunford Schwartz. p. 2162.

To apply this theorem, we must be able to find conditions on A and $\delta(A)$ for $T = \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix}$ to satisfy the two conditions above.

LEMMA 2.6. [3] *If $F = \begin{pmatrix} X & Y \\ 0 & X \end{pmatrix}$ where $X, Y \in B(\mathscr{H})$ (therefore F acts on $\mathscr{H} \oplus \mathscr{H}$), $spec(F) = spec(X)$.*

PROOF. Note that $\begin{pmatrix} X & Y \\ 0 & X \end{pmatrix}^{-1} = \begin{pmatrix} X^{-1} & -X^{-1}YX^{-1} \\ 0 & X^{-1} \end{pmatrix}$ if X is invertible.

Hence especially spec T is in real line.

LEMMA 2.7. [3] *If $\lambda \notin spec(A)$ and $A \in \mathscr{D}(\delta)$. Suppose δ is closed. Then $(\lambda I - A)^{-1} \in \mathscr{D}(\delta)$ and $\delta((\lambda I - A)^{-1}) = (\lambda I - A)^{-1}\ \delta(A)\ (\lambda I - A)^{-1}$.*

PROOF. If $\lambda \notin$ spec(A), then $1/\lambda 1 - x$ is analytic in a neighborhood of *spec*(A). Hence $(\lambda I - A)^{-1} \in \mathscr{D}(\delta)$ (see the remark after Def. 2.4. Using

$\begin{pmatrix} X & Y \\ 0 & X \end{pmatrix}^{-1} = \begin{pmatrix} X^{-1} & -X^{-1}YX^{-1} \\ 0 & X^{-1} \end{pmatrix}$, $\begin{pmatrix} \lambda I - A & \delta(\lambda I - A) \\ 0 & \lambda I - A \end{pmatrix}^{-1}$

$= \begin{pmatrix} (\lambda I - A)^{-1} & -(\lambda I - A)^{-1}\delta(\lambda I - A)(\lambda I - A)^{-1} \\ 0 & (\lambda I - A)^{-1} \end{pmatrix}$. Hence

$\delta(\lambda I - A)^{-1} = (\lambda I - A)^{-1}\delta(A)(\lambda I - A)^{-1}$ since $\delta(\lambda I - A) = \lambda\delta(I) - \delta(A) = -\delta(A)$.

LEMMA 2.8. *Let* $A = A^*$. *Then* $\|1/I - A\| < 1/|\mathrm{Im}\ \lambda|$ if $\mathrm{Im}\ \lambda \neq 0$.

PROOF. $\|(\lambda I - A)x\|^2 = \langle(\lambda I - A)x,\ (\lambda I - A)x\rangle$

$= \langle(\mathrm{Re}\ \lambda - A)x + i\ \mathrm{Im}\ \lambda x,\ (\mathrm{Re}\ \lambda - A)x + i\ \mathrm{Im}\ \lambda x\rangle$

$= \|(\mathrm{Re}\ \lambda - A)x\|^2 + |\mathrm{Im}\ \lambda x|^2$

Thus $\|x\|^2 < (\|(\lambda I - A)x\|^2)/|\mathrm{Im}\ \lambda|^2$. *Hence* $\|(\lambda I - A)^{-1}\| < 1/|\mathrm{Im}\ \lambda|$.

COROLLARY 2.9. *if* $T = \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix}$ *where* $A = A^*$, δ *as in Lemma* (2.7), *then* T *satisfies growth condition* (G_2).

PROOF. $\|(\lambda I - T)^{-1}\| \leq 4\ \max\ \{\|(\lambda I - A)^{-1}\|,\ \|(\lambda I - A)^{-1}\delta(A)(\lambda I - A)^{-1}\|\}$ $\leq M/|\mathrm{Im}\ \lambda|^2$ if $|\mathrm{Im}\ \lambda|$ is sufficiently small.

So far we obtained all the conditions in theorem 2.5 except boundedness condition. Therefore our final theorem is the following

THEOREM 2.10. *If* $A = A^* \in \mathscr{D}(\delta)$, *and* δ *is a closed derivation, then* $f(A) \in \mathscr{D}(\delta)$ *where* f *is a* C^1*-function on an open interval containing spec*(A) *if and only if* $T = \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix}$ *satisfies boundedness condition in Def.* (2.2).

COROLLARY 2.11. *Let* A, δ *be as above. If* A *and* $\delta(A)$ *commutes, then* $f(A) \in \mathscr{D}(\delta)$ *where* f *is a* C^1*-function on an open interval containing spec*(A).

PROOF. All we have to do is to show $T = \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix}$ satisfies boundedness condition.

Let $x, y \in \mathscr{H} \oplus \mathscr{H}$ be two vectors such that $\sigma_T(x) \cap \sigma_T(y) = \phi$. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ where $x_i, y_i \in \mathscr{H}$.

We are going to show that $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are orthogonal respectively, which gives boundedness condition. ($\|x\| = (\|x_1\|^2 + \|x_2\|^2)^{1/2} \leq (\|x_1 + y_1\|^2 + \|x_2 + y_2\|^2)^{1/2}$ $= \|x + x\|$.)

If $\lambda \notin \sigma_T(x)$, then we can define $(\lambda I-T)^{-1}x$

$$=\begin{pmatrix}(\lambda I-A)^{-1} & (\lambda I-A)^{-1}\delta(A)(\lambda I-A)^{-1}\\ 0 & (\lambda I-A)^{-1}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$=\begin{pmatrix}(\lambda I-A)^{-1}x_1+(\lambda I-A)^{-1}\delta(A)(\lambda I-A)^{-1}x_2\\ (\lambda I-A)^{-1}x_2\end{pmatrix}.$$

Hence $\lambda \notin \sigma_A(x_2)$ *i.e.* $\sigma_A(x_2)\subset\sigma_T(x)$. Similary $\sigma_A(y_2)\subset\sigma_T(y)$.
To take care of x_1 and y_1 we need a lemma due to Dunford-Schwartz.

LEMMA. *If P is a bounded linear operator in $\mathscr{H}$ which commutes with A, then* $\sigma_A(Px)\subset\sigma(x)$, $x\in\mathscr{H}$. See the proof in [6, p2138].—Therefore if $\lambda\notin\sigma_T(x)$ (therefore not in $\sigma_A(x_2)$) then $(\lambda I-A)^{-1}\delta(A)(\lambda I-A)^{-1}x_2=(\lambda I-A)^{-1}(\lambda I-A)^{-1}$ $\delta(A)x2$ has analytic continuation. Since $(\lambda I-A)^{-1}x_1+(\lambda I-A)^{-1}\delta(A)(\lambda I-A)^{-1}x_2$ has analytic continuation, $(\lambda I-A)^{-1}x_1$ should have analytic continuation i.e., $\lambda\notin\sigma_A(x_1)$ which is equivalent to $\sigma_A(x_1)\subset\sigma_T(x)$.

Similary $\sigma_A(y_1)\subset\sigma_T(y)$. Therefore $\sigma_A(x_1)\cap\sigma_A(y_1)\subset\sigma_T(x)\cap\sigma_T(y)=\phi$ and $\sigma_A(x_2)$ $\cap\sigma_A(x_2)\subset\sigma_T(x)\cap\sigma_T(y)=\phi$. With these results, we can follow Dunford-Schwartz argument. Consider $f_i(\lambda)=\langle R(\lambda:A)\ x_i,y_i\rangle$, $i=1,2$, then $f_i(\lambda)$ is defined if $\lambda\notin\sigma_A(x_i)$, especially if $\lambda\notin\sigma_T(x)$. We claim that f_i has analytic continuation over all complex plane by defining $f_i(\lambda)=\overline{\langle R(\bar\lambda:A)y_i,x_i\rangle}$ if $\lambda\in\sigma_A(x_i)$, which is clearly analytic and defines same function on common domain because $\langle R(\lambda:A)x_i,y_i\rangle=\langle x_i,\ R(\bar\lambda:A)y_i\rangle=\overline{\langle R(\bar\lambda:A)y_i,x_i\rangle}$.

The expansion $(R(\lambda:A)(x_i,\ y_i)=((x_i,\ y_i)/\lambda)+((Ax_i,\ y_i)/\lambda^2)+\cdots$ shows that $(R(\lambda:A)x_i y_i)$ vanishes at infinity and hence identically zero. Thus $(x_i,y_i)=0$.

REMARK. Even though we obtained a necessary and sufficient condition to our question in the title, the boundedness condition is so difficult to verify that we are able to verify this condition only in rather trivial case. So we must wait until further developments in spectral theory are achieved.

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