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LOWER FORMATION RADICAL FOR NEAR RINGS

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0. Abstract

In [7] Scott has defined C-formation radical for a class C of near rings and has studied its porperties under chain conditions. A natural question that arises is: Does there exist a Lower C-Formation radical class $L(M)$ containing a given class M of ideals of near rings in C? In this paper we answer this by giving two constructions for $L(M)$ and prove that prime radical is hereditary.

1. Introduction

Analogous to rings various type of radicals of near rings and their properties have been studied by many people namely Vander Walt, Biedleman, Laxton Ramakotaiah etc. In developing general radical theory for near rings, the problem is due to the fact that an elementwise characterization of an ideal generated by a subset is not known. Results of ring theory using ideals and homomorphisms follow easily but other usual fundamental properties that hold for associative rings, no longer hold for near rings. For example, for a radical property P (in the sense of [5, p.3]) and an ideal I of a near ring N, $P(I)$ need not be contained in $P(N)$. In [7] S.D. Scott has started an alternate general radical theoretic approach by defining a C-formation radical for a class C of near rings. Scott has shown that Baer lower radical gives rise to a C-formation radical class. In general a radical property need not give rise to a C-formation radical and vice versa. However if the radical property P is hereditary then $\bar{P}=\{(I,N)\,|\,I\subset P(N)$ is an ideal of $N\}$ becomes a C-formation radical class. In section 3 we answer the natural question of the existance of a lower C-formation radical class $L(M)$ containing a given class M of ideals of near rings in C. In [4] Levetzki has shown that for rings Baer radical [5] is equal to the intersection of prime ideals. A similar result is established for C-formation radical of a near ring. In section 4 we prove the inheritence of the hereditary property under lower C-formation radical construction, by giving another construction of $L(M)$. As a corollary it is proved that lower nil radical (here called a prime radical) defined in [2] is hereditary.

2. Preliminaries

In this paper a near ring means a left near ring satisfying $0 \cdot x=0$ for all x. For various definitions and elementary properties of a near ring we refer to [7]. Let C be a homomorphically closed class of near rings and W be the class of all ordered pairs (I, N) where $I \triangleleft N$ (I is an ideal of N) and $N \in C$. A pair (I, N) is said to be non-zero if $I \neq 0$. For a subset M of W, I is called an M-ideal of $(I, N) \in M$. A pair (J, N) in W is said to be an ideal of $(I, N) \in W$ (denoted by $(J, N) \leq (I, N)$) if $J \subset I$. The following closure operations on M are introduced in [7].

DEFINITION 2.1. A subset M of W is called *S_1-closed, Q-closed, E-closed* or *G-closed* according as $M=S_1M$, $M=QM$, $M=EM$ or $M=GM$ respectively where

(1) $S_1M=\{(J, N) \mid (J, N) \leq (I, N)$ for some pair (I, N) in $M\}$;

(2) $QM=\{(I\theta, N\theta) \mid (I, N) \in M$ and θ is a homomorphism of $N\}$;

(3) $EM=\{(I, N) \mid (I/J, N/J)$ and (J, N) are in M for some $J \triangleleft N\}$;

(4) $GM=\{(I/I \cap J, N/I \cap J) \mid ((I+J)/J, N/J) \in M$ for I, J ideals of N in $C\}$.

It is easy to see that for any subset M of $W, M^*=S_1QM$ is both S_1-closed and Q-closed.

DEFINITION 2.2. A subset P of W is said to be a *C-formation radical class* if it satisfies:

(5) $P=S_1P=QP=GP$;

(6) every $N \in C$ contains a unique maximal P-ideal $P(N)$;

(7) $(N/P(N), N/P(N)) \in SP$ for all $N \in C$ where

(8) $SP=\{(I, N) \in W \mid (I, N)$ has no non-zero ideal in $P\}$.

For a universal class C, if P is a radical class in C in the sense of [5, p, 3], then $\bar{P}=\{(I, N) \mid I \triangleleft N, N \in C$ and $I \subset P(N)\}$ has all the properties of definition 2.2 except possibly $\bar{P}=G\bar{P}$. But if P is a hereditary radical class ($P(N)=N \Longrightarrow P(I)=I$ for all $I \triangleleft N, N \in C$) then $\bar{P}$ is a C-formation radical class. Also, a C-formation radical class need not give rise to a C-radical.

In the rest of our discussion we assume that any class $M \subset W$ contains the pairs $(0, N)$ for all $N \in C$. Moreover any two pairs (I, N) and (I', N') are identical if N and N' are isomorphic and I is isomorphic to I' under the restriction map.

3. Lower C-formation radical

For the construction of lower C-formation radical class containing the given class $M(\subset W)$ we need the following characterization based on [5, Thm. 1].

THEOREM 3.1. *A subclass P of W is a C-formation radical class if and only if it satisfies properties* (5) *and*

(9) *For any $(I,N)\in W$ if $(I,N)\notin P$ then there exists a homomorphism θ of N with $I\theta\neq(0)$ such that $(I\theta,N\theta)\in SP$.*

PROOF. It suffices to show the sufficiency part. Let J be the sum of all P-ideals of N and let θ be any homomorphism of N with $J\theta\neq 0$. Then there exists a P-ideal I of N such that $I\not\subset \ker\theta$ and hence $(J\theta,N\theta)\notin SP$. But then by (9) $(J,N)\in P$. For proving (7) let $(I/J,N/J)\in P$ for some $I\lhd N$ and let θ be a homomorphism of N such that $I\theta\neq 0$. If $J\subset \mathrm{Ker}\theta$ then $(I\theta, N\theta)\in P$; otherwise $J\not\subset\ker\theta$ and $(J\theta, N\theta)\in P$ where $0\neq J\theta\subset I\theta$. Thus $(I\theta, N\theta)\notin SP$ for any homomorphism θ of N. By (9) it follows that $(I,N)\in P$.

Let $M\subset W$. Define $M_1=QM, \overline{M}_1=S_1QGM$ and for any ordinal β, $\overline{M}_\beta=S_1QGM_\beta$ where

(10) $M_\beta=\{(I,N)\mid$ for every homomorphism θ of N with $I\theta\neq 0$, $(I\theta,N\theta)$ has a non-zero ideal in $\overline{M}_\alpha$ for some ordinal $\alpha<\beta\}$.

THEOREM 3.2. $L(M)=\bigcup_\beta \overline{M}_\beta$ *is the smallest C-formation radical class (called lower C-formation radical defined by M) containing M.*

PROOF. It is easy to verify that $M_\alpha\subset\overline{M}_\alpha\subset M_\beta$ for all ordinals $\alpha<\beta$ and that $L(M)$ satisfies (5). Let $(I,N)\in W$ and let θ be a homomorphism of N such that $(I\theta, N\theta)$ has a non-zero ideal in $L(M)$. But then $(I\theta, N\theta)$ has a nonzero ideal in $\overline{M}_\alpha$ for some α and so there exists an ordinal γ such that $(I\theta,N\theta)$ has a non-zero ideal in $\overline{M}_\gamma$ for all homomorphism θ of N with $I\theta\neq 0$. Thus $(I, N)\in M_{\gamma+1}\subset L(M)$ and so (9) follows. $L(M)$ is minimal such by its construction.

An example of lower C-formation radical comes from the construction of Baer lower radical in rings [5, p. 56]. For $N\in C$ and $I\lhd N$ define I_1 to be the sum of all nilpotent ideals of N contained in I. Suppose I_α is defined for all ordinals $\alpha<\beta$. Define $I_\beta=\bigcup_{\alpha<\beta} I_\alpha$ if β is a limit ordinal. For a non-limit ordinal β, $I_\beta/I_{\beta-1}$ is the sum of all nilpotent ideals of $N/I_{\beta-1}$ contained in $I/I_{\beta-1}$. Then $B=\{(I,N)\mid N\in C, I=I_\alpha$ for some ordinal $\alpha\}$ is a C-formation radical class, called the Baer radical [7]. In fact we have the following:

THEOREM 3.3. *B is the lower C-formation radical class containing $A=\{(I,N)\mid$*

$N \in C$, I is a nilpotent ideal of $N\}$.

PROOF. Clearly $A \subset B$ and so $L(A) \subset B$. It suffices to prove that $SL(A) \subset SB$: as for any two C-formation radical classes P_1 and P_2, $P_1 \subset P_2$ if and only if $SP_2 \subset SP_1$. Consider $(I, N) \in SL(A)$ and $0 \neq (J, N) \leq (I, N)$. Then $(J, N) \in SL(A)$ and hence if $K \lhd N$ with $K \subset J$ then K is not nilpotent. Therefore $J_\alpha = 0$ for all ordinals α. This shows that $(J, N) \notin B$ and hence $(I, N) \in SB$.

The intersection of prime ideals of a ring was first considered by N. H. McCoy and only later did Levitzki show that this was equal to the Baer lower radical [4]. We motivate to prove a similar result here. The concept of prime ideals and lower nil radical was generalized to near rings by Vander Walt in [2]. By observing that "an ideal I of N is prime if and only if for all ideals J_i of $N, J_1 \cdot J_2 \cdots J_n \subset I$ implies $J_i \subset I$ for some i", it is easily seen that $1(N) = \cap$ {prime ideals of $N\} = \cap \{Q_i | N/Q_i$ has no non-zero nilpotent ideal} even in near rings. Thus $1(N) \subset B(N)$. Let I be any ideal of N and I_α's as defined earlier. It is clear that $I_1 \subset 1(N)$. Assume that for a given ordinal β, $I_\alpha \subset 1(N)$ for all $\alpha < \beta$. In case β is a limit ordinal $I_\beta = \bigcup_{\alpha<\beta} I_\alpha \subset 1(N)$, otherwise I_β is the sum of all ideals J of N, such that $I \supset J \supset I_{\beta-1}$ and $J^k \subset I_{\beta-1} \subset 1(N)$ for some positive integer k. Thus J and hence I is contained in $1(N)$. Hence by transfinite induction we have proved:

THEOREM 3.4. *$B(N) = 1(N) = \cap$ {prime ideals of N} for all $N \in C$.*

As an easy consequence we have the following:

COROLLARY 3.5. *The class $P = \{(I, N) | N \in C, I \subset 1(N)\}$ is the lower C-formation radical class containing the class $A = \{(I, N) | N \in C$ and I is a nilpotent ideal of $N\}$.*

In associative rings [5, p. 125] a radical class P is hereditary if and only if $P(R) \cap I \subset P(I)$ for all R in the universal class and $I \lhd R$. In view of this, define a class $N \subset W$ to be *hereditary if $M = HM$* where

(11) $HM = \{(I, U) | (I, N) \in M,\ U \in C$ and $I \subset U \lhd N\}$.

It is easy to verify that a C-formation radical class P is hereditary if and only if $P(N) \cap I \subset P(I)$ for all $I \lhd N$, $N \in C$. Here onward we shall assume that C is homomorphically closed as well as hereditary though the work can be carried over non-hereditary class C also. In that case one has to consider C-ideals of N only.

4. Another construction for $L(M)$

In case of associative rings, the hereditary property is inherited under lower radical construction [1], [6]. The inheritence of hereditary property under lower C-formation radical construction follows under some additional conditions.

THEOREM 4.1. *Let M be a S_1-closed and Q-closed subclass of W. If M is hereditary then so is $L(M)$.*

Before we give another construction for $L(M)$ and prove the required lemmas thereafter for the proof of this theorem, we need following two extensions of theorems in [6].

THEOREM 4.2. *A subclass P of W is a C-formation radical class if and only if P is E-closed and satisfies* (5) *and*

(12) *For any chain $\{I_n\}_{n\in\Gamma}$ of P-ideals of N, $\bigcup_n I_n$ is a P-ideal of N.*

PROOF. We only need to show 'if part' as 'only if part' follows easily from [7]. For any $N\in C$, by Zorn's lemma N has a maximal ideal J(say) such that $(J, N)\in P$. If $J=N$ we are done. So let $J\neq N$ and let $(I, N)\in P$ such that $I\not\subset J$. Then $((I+J)/J, N/J)\in P$ and hence $((I+J), N)\in P$ violating the maximality of J. Therefore J contains every P-ideal of N and (6) follows. The proof of (7) is immediate.

For any class $M\subset W$, define $DM=\{(I, N)\mid I=\bigcup_{n\in\Gamma} I_n$ for some chain $\{I_n\}_{n\in\Gamma}$ of M-ideals of $N\}$ and $FM=S_1QGM$. Obviously $M\subset DM$, $M\subset FM$ and $M\subset EM$. The proof of following theorem follows as [6].

THEOREM 4.3. *A subclass P of W is a C-formation radical class if and only if $P=EP=DP=FP$.*

For a given class $M(\subset W)$ define $M_1{}^*=QM$ and for any ordinal β

$$(13)\quad M_\beta{}^*=\begin{cases} EFM_{\beta-1}{}^* & \text{if } \beta \text{ is not a limit ordinal;} \\ D(\bigcup_{\alpha<\beta} FM_\alpha{}^*) & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

It is easy to verify that $M_\alpha{}^*\subset FM_\alpha{}^*\subset M_\beta{}^*$ and FM^* is S_1-closed as well as Q-closed for all ordinals $\alpha<\beta$. Using theorem 4.3 we get

THEOREM 4.4. *$L^*(M)=\bigcup_\beta FM_\beta{}^*$ is a C-formation radical class and $L^*(M)=L(M)$.*

We note that if M is hereditary then so are QM, EM and DM. The following lemma gives information about FM to justify the inheritence of hereditary property.

LEMMA 4.5. *For a S_1-closed subclass M of W, FM is G-closed.*

PROOF. We first note that $FM=\{(K/J, N/J)\mid(K/J, N/J)\leq((I+J)/J, N/J)$ where $I\cap U\subset J$ and $((I+U)/U, N/U)\in M$ for some ideals I, U and J of $N\}$. It is easy to verify that if $(K/J, N/J)\leq((I+J)/J, N/J)$ then there exists an ideal V of N contained in I such that $K/J=(V+J)/J$. Now let K be an ideal of N such that $((K+J)/J, N/J)$ is in FM. Then there exist ideals I and U of N such that $K\subset I, I\cap U\subset J$, $((K+J)/J, N/J)\leq((I+J)/J, N/J)$ and $((I+U)/U, N/U)\in M$. But then $((K+U)/U, N/U)\in M$ and so $((K+V)/V, N/V)\in FM$ for all ideals V of N with $K\cap U\subset V$. Since $I\cap U\subset J, K\cap U\subset K\cap J$ and hence $(K/K\cap J, N/K\cap J)\in FM$. This completes the proof.

LEMMA 4.6. *If a subclass M of W is S_1-closed and Q-closed then FM is hereditary whenever M is so.*

PROOF. Consider $(I, N)\in FM$ and $I\subset U\lhd N$. Then there exist N' in C, I', J', K', U' ideals of N' such that $(I, N)=((K'+U')/U', N'/U')\leq((I'+U')/U', N'/U')$, $I'\cap J'\subset U'$ and $((I'+J')/J', N'/J')\in M$. Thus $N=N'/U'$ (up to isomorphism). Since $U\lhd N$ we have $U=V'/U'$ for some $V'\lhd N'$. But M is hereditary and so $((I'+J')/J', (V'+I'+J')/J')\in M$. Also since $M=S_1M=QM=HM$ a careful check shows that $((K'+(J'\cap V'))/J'\cap V', V'/J'\cap V')\in M$. Therefore $(K'/K'\cap J'\cap V', V'/K'\cap J'\cap V')\in FM$ where $K'\cap J'\cap V'=K'\cap J'\subset I'\cap J'\subset U'$. But FM is Q-closed hence $((K'+U')/U', V'/U')=(I, U)\in FM$.

LEMMA 4.7. *For any class $M(\subset W)$ and any ordinal α, $M_\alpha{}^*=S_1M_\alpha{}^*=QM_\alpha{}^*$.*

PROOF. Let β be a limit ordinal. Consider $(J, N)\leq(I, N)$ where $(I, N)\in M_\beta{}^*$. Then $I=\bigcup_{n\in\Gamma} I_n$ where $\{I_n\}_{n\in\Gamma}$ is a chain of ideals of N such that $(I_n, N)\in FM_{\alpha_n}{}^*$ for some $\alpha_n<\beta$. Thus $J=\bigcup_{n\in\Gamma}\{I_n\cap J\}$ where $\{I_n\cap J\}_{n\in\Gamma}$ is a chain. Moreover, by hypothesis of transfinite induction $(I_n\cap J, N)\in FM_{\alpha_n}{}^*$ which shows that $(J, N)\in M_\beta{}^*$. Also if θ is a homomorphism of N then $FM_{\alpha_n}{}^*=QFM_{\alpha_n}{}^*$ implies $(I_n\theta, N\theta)\in FM_{\alpha_n}{}^*$ and $I\theta=\bigcup_{n\in\Gamma} I_n\theta$. Again by induction $(I\theta, N\theta)\in M_\beta{}^*$.

On the other hand if β is not a limit ordinal, $(J, N)\leq(I, N)$ where $(I, N)\in M_\beta{}^*$ then there exists $(K, N)\leq(I, N)$ such that $(I/K, N/K)$ and (K, N) are in $FM_{\beta-1}{}^*$. But then $((J+K)/K, N/K)$ and $(J\cap K, N)$ are in $FM_{\beta-1}{}^*$. By Lemma 4.5, $(J/K\cap J, N/K\cap J)\in FM_{\beta-1}{}^*$ and so $(J, N)\in M_\beta{}^*$. If θ is a homomorphism of N with $U=\text{Ker }\theta$ then $((I+U)/K+U,\ N/K+U)\in FM_{\beta-1}{}^*$ by induction hypotheses. Hence $(I\theta/K\theta,\ N\theta/K\theta)$ and $(K\theta,\ N\theta)$ are in $FM_{\beta-1}{}^*$ claiming that $(I\theta, N\theta)\in EFM_{\beta-1}{}^*=M_\beta{}^*$.

The proof of theorem 4.1 follows by using transfinite induction and Lemmas 4.6 and 4.7.

COROLLARY 4.8. *Prime radical is hereditary.*

PROOF. Since the class $A=\{(I,N)|I \text{ is nilpotent}\}$ is S_1-closed, Q-closed and hereditary, the class $B=L(A)$ is hereditary. Theorem 3.4 completes the proof.

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REFERENCES

[1] A.E. Hoffman and W.G. Leavitt, *Properties inherited by lower radical*, Pourtugalic Mathematica 27(1968), 63—66.

[2] A.P.J. Vander Walt, *Prime ideals and nil radical in near rings*, Arch. Math. 15 (1964), 408—414.

[3] A. Sulinski, R. Anderson and N.J. Divinski, *Lower radical properties for associative and alternative rings*, J. Lond. Math. Soc. 41(1966), 417—429.

[4] Jakob Levitski, *Prime ideals and the lower radical*, Amer. J. Math. 73(1951), 25—29.

[5] N.J. Divinski, *Rings and Radicals*, University of Toronto Press, Toronto(1965).

[6] R. Tangman and D. Kreiling, *Lower radicals in nonassociative rings* J. Aust. Math. Soc. 14 (1972), 419—423.

[7] S.D. Scott, *Formation radicals for near rings*, Proc. Lond. Math. Soc. 25(1972), 441—464.