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SWIP LOOPS AND GROUPOIDS

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1. Introduction

According to Professor Osborn [5], a loop $Q(\cdot)$ with identity e is called a loop with *WIP* (weak inverse property) if whenever three elements x, y, z of Q satisfy the relation $xy\cdot z=e$, they also satisfy the relation $x\cdot yz=e$. Let ρ and λ denote the right inverse operator and left inverse operator respectively. Then either of the properties $y\cdot(xy)^{\rho}=x^{\rho}$ or $(xy)^{\lambda}\cdot x=y^{\lambda}$ is equivalent to the above definition of *WIP* loops. For algebraic properties of *WIP* loops see [1] and [5].

In the present paper is considered a special class of *WIP* loops, in which the relation

(1) $$x^{\lambda}=x^{\rho}=x^{-1} \text{ (say)}$$

also holds and call them *SWIP* (special weak inverse property) loops. (1) also implies the relation

(2) $$(x^{-1})^{-1}=x.$$

The object of this paper is to give a characterization of the variety of *SWIP* loops as a subvariety of groupoids with a single identity. Similar theorems for *CWIP* (commutative weak inverse property) loop are also proved. Examples of finite *SWIP* loops and *CWIP* loops are given. These theorems are the generalization of the results due to Higman and Neumann [4], Padmanabhan [7], Sharma [2,3] and Kannappan [6] for groups, abelian groups, inverse loops, commutative inverse property loops, crossed-inverse loops and *WIP* loops respectively.

2. We say that a groupoid $Q(\cdot)$ is an *iso-SWIP loop* provided that there is a *SWIP* loop $Q(\cdot)$ which is a principal isotope of $Q(\cdot)$ such that $(\cdot)$ and $(\circ)$ are connected by either of the relations

(3) $$x\cdot y=x\circ y^{-1} \quad \text{for all } x,\ y\in Q \text{ or}$$

(4) $$x\cdot y=x^{-1}\circ y \quad \text{for all } x,\ y\in Q.$$

3. THEOREM 1. *A groupoid* $Q(\cdot)$ *is an iso-SWIP loop if and only if the identity*

(5) $$y=(uu)\cdot[(x\cdot(tt))\cdot(yx)\cdot(vv))]$$

holds for all $x, y, u, v, t \in Q$.

PROOF. Suppose the groupoid $Q(\cdot)$ satisfies the identity (5). First of all we show that $(\cdot)$ is right cancellative. Let $r\cdot a=s\cdot a$ for some $a\in Q$. Taking $x=a$ and $y=r$ in (5), we get

$r=(uu)\cdot[(a\cdot(tt))\cdot(ra)\cdot(vv))]=(uu)\cdot[(a\cdot(tt))\cdot((sa)\cdot(vv)]=s.$

Thus $(\cdot)$ is right cancellative. Keeping x, y, v, t the same and changing u to $\acute{u}$ in (5) and using the right cancellative property, we have

(6) $$uu=\acute{u}\acute{u}=\text{contant}=e \text{ (say), for all } u, \acute{u}\in Q.$$

On using (6) in (5),

(7) $$y=e\cdot[(xe)\cdot(yx\cdot e)] \text{ for all } x, y\in Q.$$

In (7), $y=e$ gives, by repeated use of (6) and the right cancellativity of $(\cdot)$,

(8) $$x=e\cdot x \text{ for all } x\in Q.$$

On using (8) in (7), it gives

(9) $$y=(xe)\cdot(yx\cdot e) \text{ for all } x, y\in Q.$$

Putting $y=x$ in (9) and using (6), we have

(10) $$x=(xe)\cdot e \text{ for all } x\in Q.$$

Let $a\cdot r=a\cdot s$. Setting $y=a$, $x=r$ and $y=a$, $x=s$ in (9), we get

$$(re)\cdot(ar\cdot e)=(se)\cdot(as\cdot e),$$

from which and the right cancellativity of $(\cdot)$, we get left cancellativity of $(\cdot)$. Further we define the operation $(\circ)$ as follows.

(11) $$x\circ y=x^{-1}\cdot y \text{ for all } x, y\in Q \quad \text{and}$$

(12) $$x^{-1}=x\cdot e \text{ for all } x\in Q.$$

Also $(x^{-1})^{-1}=(x\cdot e)^{-1}=(xe)\cdot e=x$ by (10). Thus

(13) $$(x^{-1})^{-1}=x \text{ for all } x\in Q.$$

On using (13) in (11), it gives

(14) $$x^{-1}\circ y=x\cdot y \text{ for all } x, y\in Q.$$

Further $x\circ e=x^{-1}\cdot e=(xe)\cdot e=e$ by (10)

and $e\circ x=e\cdot x=x$ by (8).

Thus e is the identity of $Q(\circ)$. The equations $a\cdot x=b$ and $y\cdot a=b$ have unique solutions in the groupoid $Q(\cdot)$. Thus, from (11) it follows that the equations $a\circ x=b$ and $y\circ a=b$ have unique solutions in the system $Q(\circ)$.

In view of (8), (12) and (14), the equation (5) can be written

(15) $$y^{-1}=x^{-1}\cdot(y^{-1}\cdot x)^{-1}=x\circ(y^{-1}\cdot x)^{-1} \text{ by (11)}$$
$$=x\circ(y\circ x)^{-1} \text{ by (11)}.$$

Thus we have proved that Q(∘) is a *SWIP* loop. In other words Q(·) is an iso-*SWIP* loop.

Conversely, let Q(·) be an iso-*SWIP* loop and let Q(∘) be the corresponding *SWIP* loop with identity e such that (·) and (∘) are connected by

(11) $$x\circ y=x^{-1}\cdot y \quad \text{for all } x,\ y\in Q.$$

Since *SWIP* loop Q(∘) satisfies (2), thus from (11), we get

(14) $$x^{-1}\circ y=x\cdot y \quad \text{for all } x,\ y\in Q.$$

Putting $y=e$ in (14), it gives

(12) $$x^{-1}=x\cdot e \quad \text{for all } x\in Q.$$

Putting $x=y$ in (14), it gives

(6) $$x\cdot x=e \quad \text{for all } x\in Q.$$

Putting $x=e$ in (11)

(8) $$y=e\cdot y \quad \text{for all } y\in Q.$$

We can easily obtain (5) by using (6), (8), (11), (12) and (14). This completes the proof of the theorem.

REMARK 1. The variety we have characterized can also be obtained from the identity

(16) $$y=[((vv)\cdot(xy))\cdot((tt)\cdot x)]\cdot(ww).$$

Let $w=w(x_1, \cdots, x_n)$ be some word in the variables $x_1, \cdots, x_n$ in the groupoid Q(·).

THEOREM 2. *The groupoid Q(·) is an iso-SWIP loop in which the law*

$$w(x_1, \cdots, x_n)=e$$

holds if and only if it satisfies the law

(17) $$y=((uu)\cdot w)\cdot[(x\cdot(tt))\cdot(yx\cdot(vv))]$$

for all $x,\ y,\ u,\ v,\ t\in Q$.

PROOF. The sufficient part is an easy consequence of Theorem 1; we need prove only the necessary part. As in the proof of Theorem 1, here we can show that (·) is right cancellative and hence for all $r,\ s\in Q$, we have

(18) $$(rr)\cdot w=(ss)\cdot w,$$

which in turn implies that

(19) $$r\cdot r=e \text{ (constant) for all } r\in Q.$$

Now putting $x=y=e$ in (17) and using (19), we get

(20) $$e\cdot w=e,$$

which by virtue of (19) give $e=w$.

The given identity (17) reduces to the identity (5) of Theorem 1 and so the groupoid $Q(\cdot)$ is an iso-*SWIP* loop, in which the identity $w=e$ is satisfied. This completes the proof of the theorem.

REMARK 2. The variety we have characterized above can also be obtained from the identity

(21) $$y=[((vv)\cdot(xy))\cdot((tt)\cdot x)]\cdot(w\cdot(uu)).$$

5. In this section we state the correspoeding theorems for *CWIP* loops. The proofs can be constructed by proceeding on the same lines as in Theorems 1 and 2.

THEOREM 3. *A groupoid $Q(\cdot)$ is an iso-CWIP loop if and only if the identity*

(22) $$y=(uu)\cdot[((tt)\cdot(y\cdot(zx)))\cdot(((rr)\cdot(x)\cdot((ss)\cdot z))]$$

for all $x, y, z, u, t, r, s\in Q$.

REMARK 3. The variety we have characterized can also be obtained from the identity

(23) $$y=[((z\cdot(ss))\cdot(x\cdot(rr)))\cdot(((zx)\cdot y)\cdot(tt))]\cdot(uu).$$

THEOREM 4. *The groupoid $Q(\cdot)$ is an iso-CWIP loop in which the law*

$$w(x_1, \cdots, x_n)=e$$

holds if and only if it satisfies the law

(24) $$y=((uu)\cdot w)\cdot[((tt)\cdot(y\cdot(zx)))\cdot(((rr)\cdot x)\cdot((ss)\cdot z))]$$

for all $x, y, z, u, t, r, s\in Q$.

REMARK 4. The variety we have characterized can also be obtained from the identity

(25) $$y=[((z\cdot(ss))\cdot(x\cdot(rr)))\cdot(((xz)\cdot y)\cdot(tt))]\cdot(w\cdot(uu)).$$

6. In this section we give examples of finite *SWIP*-loop and finite-*CWIP*-loop. The loops given by multiplication tables 1 and 2 are *SWIP*-loop and *CWIP*-loop respectively.

Table 1

	e	x	x^2	y	y^2	y^3	xy	xy^2	xy^3	x^2y	x^2y^2	x^2y^3
e	e	x	x^2	y	y^2	y^3	xy	xy^2	xy^3	x^2y	x^2y^2	x^2y^3
x^2	x	x^2	e	xy	xy^2	xy^3	x^2y	x^2y^2	x^2y^3	y	y^2	y^3
x^2	x^2	e	x	x^2y	x^2y^2	x^2y^3	y	y^2	y^3	xy	xy^2	xy^3
y	y	x^2y	xy	y^2	y^3	e	x^2	x^2y^3	x^2y^2	x	xy^3	xy^2
y^2	y^2	xy^2	x^2y^2	y^3	e	y	xy^3	x	xy	x^2y^3	x^2	x^2y
y^3	y^3	x^2y^3	xy^3	e	y	y^2	x^2y^2	x^2y	x^2	xy^2	xy	x
xy	xy	y	x^2y	x	xy^3	xy^2	e	y^3	y^2	x^2	x^2y^3	x^2y^2
xy^2	xy^2	x^2y^2	y^2	xy^3	x	xy	x^2y^3	x^2	y	y^3	e	x^2y
xy^3	xy^3	y^3	x^2y^3	xy^2	xy	x	y^2	y	e	x^2y^2	x^2y	x^2
x^2y	x^2y	xy	y	x^2	x^2y^3	x^2y^2	x	xy^3	xy^2	e	y^3	y^2
x^2y^2	x^2y^2	y^2	xy^2	x^2y^3	x^2	x^2y	y^3	e	y	xy^3	x	xy
x^2y^3	x^2y^3	xy^3	y^3	x^2y^2	x^2y	x^2	xy^2	xy	x	y^2	y	e

Table 2

	e	y	x	x^2	xy	x^2y
e	e	y	x	x^2	xy	x^2y
y	y	e	xy	x^2y	x	x^2
x	x	xy	x^2	y	x^2y	e
x^2	x^2	x^2y	y	x	e	xy
xy	xy	x	x^2y	e	x^2	y
x^2y	x^2y	x^2	e	xy	y	x

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