Kyŭngpook Math. J.
Volume 18, Number 1
June, 1978

THE ADDITIVE GROUP OF A FINITE NEAR-FIELD IS ELEMENTARY ABELIAN

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The proposition stated in the title of this note was first proved by Zassenhaus in 1936 [5]. There are several other proofs showing that a near-field has commutative addition (B. H. Neumann [4], Zemmer [6], Karzel [2]). The short proof given in this note is from the viewpoint that a near-field is a near integral domain without proper left ideals and utilizes the well-known group theoretic result of Thompson: A finite group with a fixed-point-free automorphism of prime order is nilpotent.

DEFINITION. *A near integral domain* is a (left) near-ring $(N, +, \cdot)$ such that
(1) $0x=0$ for each $x \in N$,
(2) at least one non-zero element is not a left identity,
(3) $ab=0$ implies $a=0$ or $b=0$ (no zero divisors).

Except for the trivial case of cardinality two, every near-field is a near integral domain. It is easy to show that a finite near integral domain has a fixed-point-free automorphism of prime order defined on its additive group and hence this group is nilpotent [3].

The mapping $f_c(b)=cb$ is an automorphism for each nonzero element c of a finite near integral domain. Thus characteristic subgroups of a near integral domain are left ideals. If $(N, +, \cdot)$ is a finite near integral domain without proper left ideals, as in a near-field, then since (N, +) is nilpotent and each p-Sylow subgroup is characteristic, it follows that $(N, +)$ is a p-group. The non-trivial center of $(N, +)$ is also characteristic so $(N, +)$ is an abelian p-group. Finally, the elements of order p form a characteristic subgroup and $(N, +)$ must be elementary abelian.

There are finite near integral domains without proper left ideals which are not near-fields (cf Clay [1]).

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REFERENCES

[1] J. R. Clay, *The near-rings on groups of low order*, Math. Zeitschr. 104 (1968), 364—371.

[2] H. Karzel, *Bericht über projektive Inzidenzgruppen*, Jahresber. Dt. Math. Ver. 67 (1964), 58—92.

[3] Steve Ligh, *On the additive groups of finite near integral domains and simple d.g. near-rings*, Monatsh. Math. 76(1972), 317—322.

[4] B. H. Neumann, *On the commutativity of addition*, J. London Math. Soc. 15(1940), 203—208.

[5] H. Zassenhaus, *Uber endliche Fastkörper*, Abh. Math. Sem. Univ. Hamburg, 11 (1936), 187—220.

[6] J. L. Zemmer, *The additive group of an infinite near-field is abelian*. J. London Math. Soc., 44(1969), 65—67.