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A NOTE ON THE MATRIX EXPONENTIAL

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1. Introduction.

Let A be an $n\times n$ matrix of complex constants. Then it is well-known that a fundamental matrix for the linear homogeneous system

$$\dot{x}=Ax$$

where x is $n\times n$ matrix parametrized by a variable t, is given by the matrix exponential

$$e^{At}=I+\sum_{k=1}^{\infty}t^k A^k/k!$$

Thus to solve the initial-value problem of the linear system, we need to calculate the function e^{At}. In [2], this is done via the Jordan canonical form of A. This procedure is clear at least in theory, but it is not easy to carry out the actual computation. E. J. Putzer [4] obtained some formulas for calculating e^{At}. Those formulas are based on the fact that e^{At} is an infinite polynomial in A whose coefficients are scalar functions of t that can be determined recursively by solving a simple auxiliary linear system. R. B. Kirchner [3] developed another algebraic method for computing e^{At} in terms of A and the factorization of the characteristic polynomial of A. Both methods are useful in practice and are valid for all square matrices A. However, general methods often have the disadvantage that they are not the simplest methods for certain special cases. T. M. Apostol [1] pointed out this fact and listed some explicit formulas for the polynomial e^{At} which can be obtained very easily in certain cases. While the properties of e^{At} might be obtained by referring to the known structure of A, the algebraic structure of A might be ascertained conversely by analyzing the behavior of e^{At}. A. D. Ziebur [5] proceeded in this opposite direction. We shall present the explicit formulas for computing e^{At} and the structure of A in the spirit of references [1], [3], [4], and [5].

2. General Methods.

Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A. These are not necessarily distinct.

Then

THEOREM 2.1.

$$e^{At}=\sum_{j=0}^{n-1} r_{j+1}(t)P_j$$

where $P_0=I,\ P_j=\prod_{k=1}^{j}(A-\lambda_k I)\quad j=1,2,\cdots,n$

and $r_1(t),\cdots,r_n(t)$ *is the solution of the triangular system*

$$\dot{r}_1=\lambda_1 r_1 \qquad r_1(0)=1$$

$$\dot{r}_j=r_{j-1}+\lambda_j r_j \qquad r_j(0)=0 \qquad j=2,\cdots,n$$

Proof. Define $r_0(t)=0$ and

$$F(t)=\sum_{j=0}^{n-1} r_{j+1}(t)P_j$$

Then we have, after collecting terms in r_j,

$$\dot{F}-\lambda_n F=\sum_{j=0}^{n-2}[P_{j+1}+(\lambda_{j+1}-\lambda_n)P_j]r_{j+1}$$

Using $P_{j+1}=(A-\lambda_{j+1}\ I)P_j$,

$$\dot{F}-\lambda_n F=(A-\lambda_n I)(F-r_n P_{n-1})$$

$$=(A-\lambda_n I)F-r_n P_n$$

But $P_n=0$ by the Cayley-Hamilton theorem, so $\dot{F}=AF$. Since $F(0)=I$, we have $F(t)=e^{At}$.

Let $\lambda_1,\cdots,\lambda_k$ denote the distinct eigenvalues of A. Then the characteristic polynomial of A is

$$p(\lambda)=\det(\lambda I-A)=\prod_{i=1}^{k}(\lambda-\lambda_i)^{m_i}$$

Since $p(D)\ e^{At}=p(A)\ e^{At}$, where D is the differential operator d/dt, each of the n^2 components of the matrix e^{At} satisfies the scalar differential equation

$$p(D)y=\prod_{i=1}^{k}(D-\lambda_i)^{m_i}y=0$$

Since we know the general solution of this differential equation, each element of e^{At} is a linear combination of

$$e^{\lambda_1 t},\ t\ e^{\lambda_1 t},\cdots,t^{m_1-1}\ e^{\lambda_1 t},\cdots,\ t^{m_k-1}\ e^{\lambda_k t}$$

Therefore there exist a set $M_{i,j}$ of $n\times n$ matrices of complex constants such that

(2.1) $$e^{At}=\sum_{i=1}^{k}\sum_{j=0}^{m_i-1} t^j e^{\lambda_i t} M_{i,j}$$

It can be shown that $(e^{At})'=A\,e^{At}$ if

$$\lambda_i M_{i,j}+(j+1)M_{i,j+1}=AM_{i,j} \qquad 0\leq j<m_i-1$$

$$\lambda_i M_{i,m_i-1}=AM_{i,m_i-1}$$

or, if $$M_{i,j+1}=(A-\lambda_i I)M_{i,j}/(j+1) \qquad 0\leq j<m_i-1$$

$$(A-\lambda_i I)M_{i,m_i-1}=0$$

We see that these conditions are satisfied if

$$(A-\lambda_i I)^{m_i}M_{i,0}=0$$

From the view of the Calyley-Hamilton theorem, we must have

$$M_{i,0}=p_i(A)=\prod_{j\neq i}(A-\lambda_j I)^{m_j}$$

Since, from (2.1), $I=\sum_{i=1}^{k}M_{i,0}=\sum_{i=1}^{k} p_i(A)$, it follows that $e^{At}=G(0)^{-1}G(t)$,

where $$G(t)=\sum_{i=1}^{k}\sum_{j=0}^{mi-1}p_i(A)(A-\lambda_i I)^j t^j e^{\lambda_i t}/j!$$

Thus we have the following

THEOREM 2.2. $e^{At}=G(0)^{-1}\sum_{i=1}^{k}\sum_{j=0}^{mi=1}p_i(A)(A-\lambda_i I)^j t^j e^{\lambda_i t}/j!$

where $G(0)=\sum_{i=1}^{k}p_i(A)$.

3. Special cases.

If the eigenvalues of A are either all equal or all distinct, then e^{At} can be obtained very easily. Also, when A has two distinct eigenvalues, one of which has multiplicity 1, e^{At} can be computed much more simply than the general methods in Theorems 2.1 and 2.2. We state these results in the following.

THEOREM 3.1. *If A is an $n\times n$ matrix with all its elgenvalues equal to λ then we have*

$$e^{At}=e^{\lambda t}\sum_{k=0}^{n-1}t^k(A-\lambda I)^k/k!$$

Proof. Since the matrices λtI and $(A-\lambda I)t$ commute, we have

$$e^{At}=e^{\lambda tI}e^{(A-\lambda I)t}=(e^{\lambda t}\ I)\sum_{k=0}^{\infty}t^k(A-\lambda I)^k/k!$$

The Cayley-Hamilton theorem implies that $(A-\lambda I)^k=0$ for $k\geqslant n$, so the theorem is proved.
This proof seems to be the simplest and most natural way to derive the result.

THEOREM 3.2. *If A is an $n\times n$ matrix with n distinct eigenvalues $\lambda_1,\cdots,\lambda_n$, then we have*

$$e^{At}=\sum_{k=1}^{n}e^{\lambda_k t}L_k(A)$$

where the $L_k(A)$ are Lagrange interpolation coefficients given by

$$L_k(A)=\prod_{\neq k}(A-\lambda_j I)/(\lambda_k-\lambda_j), \qquad k=1,2,\cdots,n$$

Proof. Define a matrix-valued function of the scalar t by the equation

$$F(t)=\sum_{k=1}^{n}e^{\lambda_k t}L_k(A) \tag{3.1}$$

To prove that $F(t)=e^{At}$, it suffices to show that F satisfies $F'(t)=AF(t)$, $F(0)=I$. From (3.1), we see that

$$A\ F(t)-F'(t)=\sum_{k=1}^{n}e^{\lambda_k t}(A-\lambda_k I)\ L_k(A)$$

By the Cayley-Hamilton theorm, we have $(A-\lambda_k I)L_k(A)=0$ for each k. Thus we have $F'(t)=A\ F(t)$. Also, from (3.1),

$$F(0)=\sum_{k=1}^{n}L_k(A)=I$$

which completes the proof.

THEOREM 3.3. *Let A be an $n\times n$ matrix $(n\geqslant 3)$ with two distinct eigenvalues λ and μ, where λ has multiplicity $n-1$ and μ has multiplicity 1. Then we have*

$$e^{At}=e^{\lambda t}\sum_{k=0}^{n-2}t^k(A-\lambda I)^k/k!$$
$$+\{e^{\mu t}/(\mu-\lambda)^{n-1}-e^{\lambda t}/(\mu-\lambda)^{n-1}\sum_{k=0}^{n-2}t^k(\mu-\lambda)^k/k!\}(A-\lambda I)^{n-1}$$

Proof. As in the proof of Theorem 3.1, we have

$$e^{At}=e^{\lambda t}\sum_{k=0}^{\infty}t^k(A-\lambda I)^k/k!$$
$$=e^{\lambda t}\sum_{k=0}^{n-2}t^k(A-\lambda I)^k/k!+e^{\lambda t}\sum_{k=n-1}^{\infty}t^k(A-\lambda I)^k/k!$$

$$=e^{\lambda t}\sum_{k=0}^{n-2}t^k(A-\lambda I)^k/k!+e^{\lambda t}\sum_{r=0}^{\infty}t^{n-1+r}(A-\lambda I)^{n-1+r}/(n-1+r)!$$

Now we evaluate the series over r in closed form by using the Cayley-Hamilton theorem. Since $A-\mu I=A-\lambda I-(\mu-\lambda)I$, we find $(A-\lambda I)^{n-1}(A-\mu I)=(A-\lambda I)^n-(\mu-\lambda)(A-\lambda I)^{n-1}$. The left member is 0 by the Cayley-Hamilton theorem, and thus

$$(A-\lambda I)^n=(\mu-\lambda)(A-\lambda I)^{n-1}$$

Using this relation repeatedly, we have

$$(A-\lambda I)^{n-1+r}=(\mu-\lambda)^r(A-\lambda I)^{n-1}$$

Therefore the series over r becomes

$$\sum_{r=0}^{\infty}t^{n-1+r}\ (\mu-\lambda)^r\ (A-\lambda I)^{n-1}/(n-1+r)!$$

$$=\sum_{k=n-1}^{\infty}t^k(\mu-\lambda)^k(A-\lambda I)^{n-1}/k!\,(\mu-\lambda)^{n-1}$$

$$=\left\{e^{(\mu-\lambda)t}-\sum_{k=0}^{n-2}t^k(\mu-\lambda)^k/k!\right\}(A-\lambda I)^{n-1}/(\mu-\lambda)^{n-1}$$

which completes the proof.

Note that the explicit formulas in Theorems 3.1, 3.2, and 3.3 cover all matrices of order $n\leqslant 3$.

4. The structure of A.

Let us assume that

(4.1) $$M_{i,j}=0 \text{ for } j\geqslant m_i$$

Then we can write (2.1) in the following form

(4.2) $$e^{At}=\sum_{i=1}^{k}\sum_{j=0}^{n}t^je^{\lambda_it}M_{i,j}$$

Thus, for arbitrary numbers r and s, we have

(4.3) $$e^{Ar}e^{As}=\sum_{\rho,\xi=1}^{k}\sum_{\sigma,\eta=0}^{n}r_\sigma s^\eta e^{\lambda_\rho r+\lambda_\xi s}M_{\rho,\sigma}M_{,\eta}$$

Also, from (4,2), we have

$$e^{A(r+s)}=\sum_{i=1}^{k}\sum_{j=0}^{n}(r+s)^je^{\lambda_i(r+s)}M_{i,j}$$

(4.4) $$=\sum_{\rho,\xi=1}^{k}\sum_{j=0}^{n}(r+s)^j\delta_{\rho\xi}e^{\lambda_\rho r+\lambda_\xi s}M_{\rho,j}$$

$$= \sum_{\rho,\xi=1}^{k} \sum_{\sigma,\eta=0}^{n} \delta_{\rho\xi} \binom{\sigma+\eta}{\sigma} r^{\sigma} s^{\eta} e^{\lambda_\rho r+\lambda_\xi s} M_{\rho,\sigma+\eta}$$

Since the left-hand sides of (4.3) and (4.4) are equal and since the λ's are distinct and r and s are arbitrary, we can equate the coefficients of $r^{\sigma}s^{\eta}\cdot e^{\lambda_\rho r+\lambda_\xi s}$ and obtain the following basic relation among the matrices of the set $M_{i,j}$

(4.5) $$M_{\rho,\sigma} M_{\xi,\eta} = \delta_{\rho\xi} \binom{\sigma+\eta}{\sigma} M_{\rho,\sigma+\eta}$$

If we fix our attention on some particular index ρ and set $\xi=\rho$ and $\sigma=\eta=0$, then (4.5) says that $M_{\rho,0}{}^2=M_{\rho,0}$. That is $M_{\rho,0}$ is a projection matrix. Still letting $\xi=\rho$, let σ be arbitrary, and set $\eta=1$ to obtain the equation $M_{\rho,\sigma+1}=(\sigma+1)^{-1} M_{\rho,\sigma} M_{\rho,1}$. We use this recursion formula and mathematical induction to establish the formula

(4.6) $$M_{\rho,\sigma}=M^{\sigma}{}_{\rho,1}/\sigma! \qquad \sigma \geqslant 1$$

In particular, since $M_{\rho,m_\rho}=0$, we see that $M_{\rho,1}{}^{m_\rho}=0$. That is, $M_{\rho,1}$ is a nilpotent matrix. Thus, all the coefficient matices can be expressed in terms of certain projection matrices and nilpotent matrices. Denoting the projection matrix $M_{\rho,0}$ by P_ρ and the nilpotent matrix $M_{\rho,1}$ ay N_ρ, the relation (4.5) gives the following basic relations

(4.7) $$\begin{gathered} P_i P_j = \delta_{ij} P_j \\ P_i N_j = N_j P_i = \delta_{ij} N_j \\ N_i N_j = \delta_{ij} N_j^2 \\ N_i^{m_i} = 0 \end{gathered}$$

From (4.6) and (4,7), we have $M_{i,j}=N_i^j P_i/j!$. Therefore the formula far e^{At} can be written

(4.8) $$e^{At} = \sum_{i=1}^{k} \sum_{j=0}^{m_i-1} t^j e^{\lambda_i t} N_i^j P_i / j!$$

Equation (4.8) may give the structure of A. First, set $t=0$. Then

(4.9) $$\sum_{i=1}^{k} P_i = I$$

This equation, tegether with the equation $P_i P_j=\delta_{ij}P_j$, tells us that the family of projections $\{P_i\}$ decomposes the complex spaces C_n into a direct sum of disjoint subspaces. Now differentiate both sides of (4.8), and set $t=0$. Then we have the desired decomposition of A, (4.10) $$A=\sum_{i=1}^{k}(\lambda_i I+N_i)P_i=\sum_{i=1}^{k}(\lambda_i P_i+N_i)$$

Equation (4.10) shows how A can be expressed in terms of certain coefficients of a power series whose sum is e^{At}. This decomposition of A is unique [5].

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