Bull. Korean Math. Soc.
Vol. 15, No. 1, 1978.

ON SOME TRANSITIVE PERMUTATION GROUPS

BY SEUNG AHN PARK

1. Introduction

The purpose of this paper is to characterize transitive permutation groups satisfying a certain condition. Our theorem is the following:

THEOREM. *Let G be a transitive permutation group on a finite set. Suppose that G contains a subgroup K such that*

(1) *K is transitive, and*

(2) *$C_G(k)=K$ for all nonidentity elements k in K.*

Then one of the following holds:

(i) *G is doubly transitive, or*

(ii) *Either $G=K$, or G is a Frobenius group with kernel K.*

As a corollary we obtain a well-known theorem of Burnside [1].

COROLLARY (Burnside). *Let G be a transitive permutation group of prime degree p. Then one of the following holds:*

(i) *G is doubly transitive, or*

(ii) *Either G is cyclic of order p, or G is a Frobenius group with kernel of order p; hence G is solvable.*

Our proof of Theorem requires some theorems on group characters and the structure theorem on Frobenius groups. A theorem on exceptional charaters proved by Brauer and Suzuki [7] will play an important role in our proof. For the general discussion on group characters we refer to [2, 3, 5, 6].

All groups in this paper are assumed to be finite. Our notation is standard and taken from [4]. If π is a set of primes, a subgroup K of a group G is called a *Hall π-subgroup* of G provided K is a π-group and $|G:K|$ is divisible by no primes in π. A group G containing a nonidentity proper subgroup H is called a *Frobenius group* with *complement* H if H is a $T.I.$ set in G and $H=N_G(H)$. A Frobenius group G with complement H contains a normal complement K to H in G, which is called the *Frobenius kernel* of G.

2. Necessary lemmas

In this section we will give two necessary lemmas.

(2. 1) *Let G be a group containing a nonidentity proper normal subgroup K such that $C_G(k)=K$ for all nonidentity element k in K. Then G is a Frobenius group with kernel K.*

Proof. This is well-known. We can show that K is an abelian Hall subgroup of G. Hence from Schur-Zassenhaus' theorem it follows that G has a complement H to K. Now it is easy to see that H is a $T.I.$ set in G and $H=N_G(H)$. Therefore, G is a Frobenius group with kernel K.

(2. 2) *If a group G has a nilpotent Hall π-subgroup, then any two Hall π-subgroups of G are conjugate.*

Proof. This is a theorem of Wielandt [8].

3. Proof of Theorem

In this sectin we will prove Theorem.

Assume that G is a transitive permutation group satisfying the hypothesis of Theorem, and assume that G is not doubly transitive. To prove Theorem it suffices to show that either $G=K$ or G is a Frobenius group with kernel K.

The proof of Theorem is divided into 9 steps.

(3. 1) *The subgroup K is an abelian Hall subgroup of G.*
If $N_G(K)\neq K$, then $N_G(K)$ is a Frobenius group with kernel K.

Proof. By assumption K is abelian. Now the assertion follows from (2. 1).

(3. 2) *The subgroup K is a $T.I.$ set in G.*

Proof. Suppose that $K\cap K^x\neq 1$ for some element x in G. Then there is a nonidentity element k in K such that $k^x\in K$. By condition (2), this implies that

$$K^x=C_G(k)^x=C_G(k^x)=K.$$

Hence K is a $T.I.$ set in G.

(3. 3) *Let θ be the permutation character of G over the complex field. Then*

$$\theta=1_G+\sum_{i=1}^{t}\chi_i,$$

where the χ_i are irreducible complex characters of G not 1_G, and $t\geq 2$.

Proof. This follows from the fact that G is transitive but not doubly transitive.

(3.4) *The restriction* θ_K *of* θ *on* K *is the sum of exactly* $|K|$ *distinct linear characters of* K. *In particular*

$$(\chi_i|_K,\ 1_K)_K=0 \text{ and } (\chi_i|_K,\ \chi_j|_K)_K=0 \ \textit{for}\ i\neq j.$$

Proof. Since K is an abelian transitive group, K is a regular permutation group. Hence θ_K is the character of the regular representation of an abelian group K. This proves (3.4).

(3.5) *The* χ_i *are all different.*

Proof. By (3.4) we have $\chi_i|_K\neq\chi_j|_K$ for $i\neq j$. Hence the assertion holds.

(3.6) *Let* λ_i *be a linear irreducible constituent of* $\chi_i|_K$ *for* $1\leq i\leq t$. *Then there exists a sign* $\varepsilon=\pm 1$ such that

$$\lambda_i^G-\lambda_j^G=\varepsilon(\chi_i-\chi_j), \text{ for all } 1\leq i, j\leq t.$$

Proof. Let $N=N_G(K)$. Then it follows from (3.1) that either $N=K$ or N is a Frobeius group with kernel K. Therefore, the λ_i^N are irreducible characters of N which vanish on $N-K$ and $\lambda_i^N(1)=|N:K|$. Since $(\lambda_i^N)^G=\lambda_i^G$ and

$$(\lambda_i^G,\ \chi_j)_G=(\lambda_i,\ \chi_j|_K)_K=\begin{cases}1 \text{ if } i=j.\\ 0 \text{ if } i\neq j,\end{cases}$$

it follows that the λ_i^N are all different. Moreover, K is a *T.I.* set in G by (3.2). Hence by a theorem of Brauer and Suzuki [3, Theorem 12.1], it follows that there exists a sign $\varepsilon=\pm 1$ and distinct irreducible complex characters $\phi_1, \phi_2, \cdots, \phi_t$ of G such that

$$\lambda_i^G-\lambda_j^G=\varepsilon(\phi_i-\phi_j) \text{ for all } 1\leq i, j\leq t.$$

On the other hand, it follows from (3.4) that

$$(\lambda_i^G-\lambda_j^G,\ \chi_i)_G=(\lambda_i-\lambda_j, \chi_i|_K)_K=1 \text{ for } i\neq j.$$

Hence we have

$$\lambda_i^G-\lambda_j^G=\pm\varepsilon(\chi_i-\chi_j),$$

and the assertion follows.

(3.7) *Let* $L=\bigcup_{g\in G}K^{*g}$ *where* $K^*=K-\{1\}$. *Then for each* x *in* $G-L$

$$\chi_1(x)=\chi_2(x)=\cdots=\chi_t(x)$$

and $\chi_1(x)$ *is a nonnegative rational integer.*

Proof. For each i, $1\leq i\leq t$, we have

$$\lambda_i^G(x)=\begin{cases}|G:K| & \text{if } x=1,\\ 0 & \text{if } x\neq 1.\end{cases}$$

Hence it follows from (3.6) that

$$\chi_i(x)-\chi_j(x)=\varepsilon(\lambda_i^G(x)-\lambda_j^G(x))=0,$$

and we have

$$\chi_i(x)=\chi_1(x) \text{ for all } i.$$

Since $\theta(x)\,(=1+\sum_{i=1}^{t}\chi_i(x)=1+t\chi_1(x))$ is a a rational integer, $\chi_1(x)$ is a rational number. The number $\chi_1(x)$ is also an algebraic integer, and hence it must be a rational integer. From the fact that $\theta(x)=1+t\chi_1(x)\geq 0$ it follows that $\chi_1(x)\geq 0$.

(3.8) *Let L be as in (3.7). For $0<i\leq s=\chi_1(1)$ let n_i be the number of elements x in $G-L$ such that $\chi_1(x)=i$. Let n be the number of conjugates of K in G. Then*

(i) $\sum_{i=1}^{s} in_i=ns$ *and* (ii) $\sum_{i=1}^{s} i^2n_i=ns^2$.

Proof. By (3.4) we have

$$0=|K|(\chi_1|_K,\ 1_K)_K=\sum_{k\in K}\chi_1(k).$$

Hence it follows from (3.7) that

$$0=|G|(\chi_1,\ 1_G)_G=\sum_{x\in G}\chi_1(x)=\sum_{x\in G-L}\chi_1(x)+\sum_{x\in L}\chi_1(x)$$

$$=\sum_{i=1}^{s} in_i+n(0-\chi_1(1))=\sum_{i=1}^{s} in_i-ns.$$

This yields the equality (i).

By (3.4) and (3.5) we have

$$0=|K|(\chi_1|_K,\ \chi_2|_K)_K=\sum_{k\in K}\chi_1(k)\chi_2(k^{-1})$$

and

$$0=|G|(\chi_1,\chi_2)_G=\sum_{x\in G}\chi_1(x)\chi_2(x^{-1}).$$

Hence it follows from (3.7) that

$$0=\sum_{i=1}^{s} i^2n_i+n(0-\chi_1(1)\chi_2(1))=\sum_{i=1}^{s} n_i-ns^2.$$

This gives the equality (ii).

(3.9) *Either $G=K$, or G is a Frobenius group with kernel K.*

Proof. From the equalities in (3.8) it follows that

$$0=(s-1)n_1+2(s-2)n_2+\cdots+i(s-i)n_i+\cdots+(s-1)n_{s-1}.$$

Hence we obtain $n_1=n_2=\cdots=n_{s-1}=0$. By the equality (i) in (3.8) this implies that $n=n_s$.

Suppose that x is an element in $G-L$ such that $\chi_1(x)=\chi_1(1)=s$. Then it follows from (3.7) that

$$\theta(x)=1+t\chi_1(x)=1+t\chi_1(1)=\theta(1),$$

which implies that $x=1$. Therefore, we have $n=n_s=1$, and K is normal in G. Now the assertion follows from (3.1).

This completes the proof of Theorem.

4. Proof of Corollary

To prove Corollary we assume that G is a transitive permutation group of prime degree p and that G is not doubly transitive.

Let K be a Sylow p-group of G. Since G is transitive, p is a divisor of $|G|$. Since G is a subgroup of the symmetric group of degree p, $|G|$ is a divisor of $p!$. Hence K is of order p. Now it is easy to see that K is generated by a p-cycle, and K is transitive and regular. This implies that $C_G(K)=K$, and $C_G(k)=K$ for all nonidentity elements k in K. Thus the group G satisfies the hypothesis of Theorem.

By Theorem either $G=K$, or G is a Frobenius group with kernel K. If $G=K$, then G is cyclic of order p, and hence G is solvable. Assume that G is a Frobenius group with kernel K. Since

$$G/K=N_G(K)/C_G(K),$$

the factor group G/K is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(K)$ of K. The automorphism group $\mathrm{Aut}(K)$ is cyclic, since K is cyclic of order p. Hence G/K is cyclic. Therefore, G is solvable.

This completes the proof of Corollary.

References

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Sogang University