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A NOTE ON THE CATEGORY End(R)

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Let R be a commutative ring with 1. We shall denote the category consisting of all R-modules and R-homomorphisms by $M(R)$. Then $M(R)$ is a complete and cocomplete category, and also it is a C_3-category with a projective generator R([3], p. 73). Therefore $M(R)$ has enough injectives ([2], p. 262). Let End(R) be the category of all R-endomorphisms, and let Idm (R) be the full subcategory of End(R) whose objects are idempotents. In [1], Hou proved that End(R) and Idm(R) have enough projectives. Let Idm-Iso(R) be the full subcategory of Idm(R) whose objects are isomorphisms. The purpose of this paper is to prove that

(i) End(R) is complete and cocomplete, and so is Idm(R) (Theorem 7 and Corollary 8),

(ii) Idm-Iso(R) has enough injectives (Theorem 11), and to prove a property about Idm(R) (Proposition 9).

1. Definitions in End(R)

It is well known that for each object α of End(R) there exists a unique object A of $M(R)$ such that α belongs to hom (A, A), where hom(A, A) is an abelian group consisiting of all R-module homomorphisms from A to itself. A morphism $f:\alpha\to\beta$ in End(R) is an R-module homomrphism such that $f\alpha=\beta f$,
where $\alpha\in$hom(A, A) and $\beta\in$hom(B, B).

DEFINITION 1. A morphism $f:\alpha\to\beta$ ($\alpha\in$hom(A, A) and $\beta\in$hom (B, B)) in End(R) is said to be a monomorphism if $f: A\to B$ is a monomorphism in $M(R)$. Dually, a morphism $f:\alpha\to\beta$ in End(R) is an epimorphism if $f: A\to B$ is an epimorphism in $M(R)$.

For a morphism $f:\alpha\to\beta(f: A\to B$ in $M(R))$ in End(R) the kernel of is defined by $\alpha|$Ker(f):Ker$(f)\to$Ker(f), where Ker(f) is the kernel of $f: A\to B$ in $M(R)$. Similarly, the image of f in End(R) is defined by $\beta|$ Im(f), where Im(f) is the image of $f: A\to B$ in $M(R)$.

The cokernel $\bar{\beta}: B/$Im$(f)\longrightarrow B/$Im(f) of $f:\alpha\to\beta$ in End(R) is defined by

$$\bar{\beta}(b+\mathrm{Im}(f))=\beta(b)+\mathrm{Im}(f)$$

for $b+\mathrm{Im}(f)\in B/\mathrm{Im}(f)$.

With the above notions we can easily prove that the categories End(R), Idm(R) and Idm-Iso(R) are abelian ([1]).

DEFINITION 2. A subobject $\beta(\beta:B\to B$ in $M(R))$ of $\alpha\in\mathrm{End}(R)$ $(\alpha:A\to A$ in $M(R))$ is an object of End(R) satisfying (i) B is a submodule of A in $M(R)$. (ii) $\beta=\alpha|B$. The intersection of subobjects β and $\gamma(\gamma:C\to C$ in $M(R))$ is $\alpha|B\cap C$. It is easy to prove that

PROPOSITION 3. *For a morphism* $f:\alpha\to\beta$ *the kernel of* f *is a subobject of* α, *and the image of* f *is a subobject of* β.

DEFINITION 4. In End(R), consider a diagram

$$\begin{array}{ccc} \alpha' & & \beta' \\ \downarrow & & \downarrow \\ \alpha & \xrightarrow{f} & \beta \end{array}$$

where f is any morphism and the vertical morphisms are monomorphisms. In this case, the subobject α' is said to be carried into the subobject β' by f if there is a morphism $\alpha'\to\beta'$ making the above diagram is commutative.

The union of a family $\{\alpha_i\}_{i\in I}$ of subobjects of an object α is defined as a subobject α' of α, denoted by $\alpha'=\cup\alpha_i$, which is preceded by each of the α_i, and which has the following property: If $f:\alpha\to\beta$ and each α_i is carried into some subobject β' by f, then α' is also carried into β' by f. In this case, if $\alpha_i:A\longrightarrow A_i$ in $M(R)$, then

$$\cup\alpha_i:\cup A_i\longrightarrow\cup A_i$$

where for $a_i\in A_i\subset\cup A_i$, $\cup\alpha_i(a_i)=\alpha_i(a_i)$.

PROPOSITION 5. *In End*(R), *for any direct family* $\{\alpha_i\}$ *of subobjects of* α *and any subobject* β *of* α

$$(\cup\alpha_i)\cap\beta=\cup(\alpha_i\cap\beta).$$

Proof. For each i we put $\alpha_i:A_i\longrightarrow A_i$, where A_i is a submodule of A in $M(R)$. By a direct family $\{\alpha_i\}$ we mean that $A_i\cap A_j=\{0\}$, if $i\neq j$. Then

$$\oplus\alpha_i=\alpha|\oplus A_i=\alpha|\cup A_i=\cup\alpha_i.$$

Since $(\cup A_i)\cap B=\cup(A_i\cap B)$ we have

$$\alpha|(\cup A_i)\cap B=\alpha|\cup(A_i\cap B)$$

and $(\cup \alpha_i) \cap \beta = \cup (\alpha_i \cap \beta)$.

PROPOSITION 6. *The category End(R) has equalizers, and so has the category Idm(R).*

Proof. If $f, g: \alpha \to \beta$ are morphisms in End(R), then in $M(R)$ we have the commutative diagrams:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha\downarrow & & \downarrow\beta \\ A & \xrightarrow{f} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{g} & B \\ \alpha\downarrow & & \downarrow\beta \\ A & \xrightarrow{g} & B \end{array}$$

Put $K = \{a \in A \mid f(a) = g(a)\}$, then K is a submodule of A. If we shall define $\gamma: K \longrightarrow K$ by $\gamma = \alpha \mid K$ then the inclusion $K \longrightarrow A$ is a morphism $\gamma \to \alpha$ in End(R). Then, it is ensily proved that $\gamma \to \alpha$ is an equalizer for f and g.

2. Main results

THEOREM 7. *End(R) is conplete and cocomplet.*

Proof. Let $\{\beta \to \alpha_i\}_{i \in I}$ be a compatible family for D. Then, for all $m \in M$ and $d(m) = (i, j)$ there is a commutative diagram:

$$\begin{array}{ccc} & \nearrow^{g_i} & A_i \\ B & & \downarrow D(m) = f_{ij} \ (f_{ij} g_i = g_j) \\ & \searrow_{g_j} & A_j \end{array}$$

such that $\alpha_j f_{ij} = f_{ij} \alpha_i$, $\alpha_i g_i = g_i \beta$ and $\alpha_j g_j = g_j \beta$ where $\beta: B \to B$ and $\alpha_i: A_i \longrightarrow A_i$ in $M(R)$. Therefore we have a compatible family $\{B \to A_i\}_{i \in I}$ for D in $M(R)$. In this case

$$\bigcap_{m \in M} \mathrm{Equ}(P_k, D(m) p_j) \subset \underset{h \in I}{\times} A_h \xrightarrow{p_i} A_i$$

is a limit for D in $M(R)$. Let us define $\underset{h \in I}{\times} \alpha_h : \underset{h \in I}{\times} A_h \longrightarrow \underset{h \in I}{\times} A_h$ such that $(\underset{h \in I}{\times} \alpha_h) \mid A_h = \alpha_h$ where $\alpha_h \in \hom(A_h, A_h)$. Then $\underset{h \in I}{\times} \alpha_h$ is a product of the family $\{\alpha_i\}_{i \in I}$ in End(R). Since

$$\bigcap_{m \in M} \mathrm{Equ}(p_k, D(m) P_j) \subset \underset{h \in I}{\times} A_h \quad ([3] \text{ p. } 47)$$

we can put $\underset{h \in I}{\times} A_h' = \bigcap_{m \in M} \mathrm{Equ}(p_k, D(m) p_j)$
where for all $h \in I$,

$$A_h' = A_h \cap (\bigcap_{m\in M} \mathrm{Equ}(p_k, D(m)p_j).$$

Define

$$\underset{h\in I}{\times}(\alpha_h | A_h') : \underset{h\in I}{\times} A_h' \longrightarrow \underset{h\in I}{\times} A_h$$

then $\underset{h\in I}{\times}(\alpha_h | A_h')$ is a limit for D in $\mathrm{End}(R)$ by the following reasons.

At first, we have the commutative diagram for all $i\in I$:

$$\begin{array}{ccc}
\underset{h\in I}{\times} A_h' & \xrightarrow{\ p_i | A_i'\ } & A_i \\
{\scriptstyle \underset{h\in I}{\times}(\alpha_h | A_h')}\downarrow & & \downarrow{\scriptstyle \alpha_i} \\
\underset{h\in I}{\times} A_h' & \xrightarrow{\ p_i | A_i'\ } & A_i
\end{array}$$

Since it is easy to see that $\alpha_i(p_i | A_i')(\underset{h\in I}{\times} a_h') = \alpha_i(a_i') = (p_i | A_i')(\underset{h\in I}{\times}(\alpha_h | A_h'))$ $(\underset{h\in I}{\times} a_h')$ for $(\underset{h\in I}{\times} a_h' \in \underset{h\in I}{\times} A_h' (a_h' \in A_h')$, and that $\{p_i | A_i' : \underset{h\in I}{\times}(\alpha_h | A_h') \longrightarrow \alpha_i\}_{i\in I}$ is a compatible family for D in $\mathrm{End}(R)$.

Next, for a compatible family $\{g_i : \beta \longrightarrow \alpha_i\}_{i\in I} (g_i : B \longrightarrow A_i)$ we have a unique R-module homomorphism $h : B \longrightarrow \underset{h\in I}{\times} A_h'$ satisfying the commutative diagram (for all $i\in I$):

$$\begin{array}{ccc}
B & \xrightarrow{\ h\ } & \underset{h\in I}{\times} A_h' \\
 & {\scriptstyle g_i}\searrow & \downarrow{\scriptstyle p_i | A_i'} \\
 & & A_i
\end{array}$$

in which $h = \underset{h\in I}{\times} g_h$, i.e. for all $b\in B$ $h(b) = \underset{h\in I}{\times} g_h(b) \in \underset{h\in I}{\times} A_h'$. Moreover the diagram

$$\begin{array}{ccc}
B & \xrightarrow{\ h = \underset{h\in I}{\times} g_h\ } & \underset{h\in I}{\times} A_h' \\
{\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \underset{h\in I}{\times}(\alpha_h | A_h')} \\
B & \xrightarrow{\ h\ } & \underset{h\in I}{\times} A_h'
\end{array}$$

is commutative since $\alpha_i g_i = g_i \beta$. It means that there exists a unique morphism $h : \beta \longrightarrow \underset{h\in I}{\times}(\alpha_h | A_h')$ such that for all $i\in I$ $g_i = (p_i | A_i')h$. For a cocompatible family $\{\alpha_i \longrightarrow \gamma\}_{i\in I}$ $(\alpha_i: A_i \longrightarrow A_i$ and $\gamma: C \longrightarrow C$ in $M(R))$ we see that $A_i \xrightarrow{u_i} \bigoplus_{h\in I} A_h \longrightarrow \bigoplus_{h\in I} A_h / \bigcup_{m\in M} \operatorname{Im}(u_k - u_j D(m))$ ([3], p. 47) is a colimit for D in $M(R)$. Put

$$\bar{A}_i = A_i / \bigcup_{m\in M} \operatorname{Im}(u_k - u_j D(m))$$

for all $i \in I (A_i \subset \bigoplus_{h\in I} A_i)$. Then

$$\bigoplus_{h\in I} A_h \bigcup_{m\in M} \operatorname{Im}(u_k - u_j D(m)) = \bigoplus_{h\in I} \bar{A}_h$$

Define

$$\bigoplus_{h\in I} \bar{\alpha}_h : \bigoplus_{h\in I} \bar{A}_h \longrightarrow \bigoplus_{h\in I} \bar{A}_h$$

such that for $a_i + \bigcup_{m\in M} \operatorname{Im}(u_k - u_j D(m)) \in \bar{A}_i$

$$(\bigoplus_{h\in I} \bar{\alpha}_h)(a_i + \bigcup_{m\in M} \operatorname{Im}\ (u_k - u_j D(m))) = \alpha_i(a_i) + \bigcup_{m\in M} \operatorname{Im}(u_k - u_j D(m)).$$

Let $\bar{u}_i: A_i \longrightarrow \bigoplus_{h\in I} \bar{A}_h$ be induced from the ith injection $u_i: A \longrightarrow \bigoplus_{h\in I} A_h$ for all $i \in I$. Then we have the commutative diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\bar{u}_i} & \bigoplus_{h\in I} \bar{A}_h \\ \downarrow \alpha_i & & \downarrow \bigoplus_{h\in I} \bar{\alpha}_h \\ A_i & \xrightarrow{\bar{u}_i} & \bigoplus_{h\in I} \bar{A}_h \end{array}$$

which means that $\bar{u}_i: \alpha_i \longrightarrow \bigoplus_{h\in I} \bar{\alpha}_h$ is a morphism of End(R).

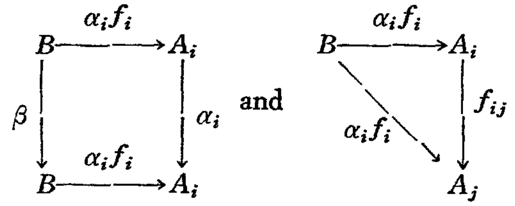
We can prove that $\{\alpha_i \rightarrow \bigoplus_{h\in I} \bar{\alpha}_h\}$ is a colimit for D by the same way as in the above proof with respect to limit. Hence End(R) is complete and cocomplete.

COROLLARY 8. *The category Idm(R) is also complete and cocomplete.*

Proof. In the category Idm(R), let $\{\beta \rightarrow \alpha_i\}_{i\in I}$ be a compatible family for a diagram over a scheme $\Sigma = (I, M, d)$ where $\alpha_i: A_i \longrightarrow A_i$ and $\alpha_i^2 = \alpha_i$ in $M(R)$ for all $i \in I$. Then its limit and colimit for D are $\times_{h\in I} (\alpha_i | A_i'$ and $\bigoplus_{h\in I} \bar{\alpha}_h$ respectively (see proof of Theorem 7), because $(\times_{h\in I} (\alpha_i | A_i'))^2 = \times_{h\in I} (\alpha_i^2 | A_i') = \times_{h\in I} (\alpha_i | A_i')$ and $(\bigoplus_{h\in I} \bar{\alpha}_h)^2 = \bigoplus_{h\in I} \bar{\alpha}_h^2 = \bigoplus_{h\in I} \bar{\alpha}_h$.

PROPOSITION 9. *In the category Idm(R), if* $\{f_i:\beta\longrightarrow\alpha_i\}_{i\in I}$ *is a compatible family, then so are* $\{\alpha_i f_i:\beta\longrightarrow\alpha_i\}_{i\in I}$ and $\{f_i\beta:\beta\longrightarrow\alpha_i\}_{i\in I}$.

Proof. For $i\in I$ we assume that $\alpha_i: A_i\longrightarrow A_i$ and $\beta: B\longrightarrow B$ in $M(R)$. For each $(i, j)\in I\times I$ we have to verify that the diagrams



are commutative. Since $\alpha_i^2=\alpha_i$, $\beta^2=\beta$, $f_{ij}f_i=f_j$, $f_i\beta=\alpha_i f_i$ and $\alpha_i f_{ij}=f_{ij}\alpha_i$ we have

$$\alpha_i\alpha_i f_i=\alpha_i f_i\beta=\alpha_i f_i \text{ and } f_{ij}\alpha_i f_i=\alpha_j f_j.$$

For $\{f_i\beta:\beta\longrightarrow\alpha_i\}$ we can prove it by the same way as above.

PROPOSITION 10. *If* $\alpha\in Idm\text{-}Iso(R)$ *then* α *is an identity map.*

Proof. By the definition of α we have $\alpha^2=\alpha$. Since α is an isomorphism there exists the inverse α^{-1} of α. Thus $\alpha^{-1}\alpha^2=\alpha^{-1}\alpha=1$.

THEOREM 11. *The category Idm-Iso(R) has enough injectives.*

Proof. Noting that the category $M(R)$ has enough injectives, it suffices to prove that $M(R)$ and Idm-Iso(R) are isomorphic.

By Propostion 9 the object class of Idm-Iso(R) is the class $\{1_A|A\in M(R)\}$.

For 1_A and 1_B in Idm-Iso(R) it is easily see that $\hom(1_A, 1_B)=\hom(A, B)$, where $\hom(1_A, 1_B)$ is the R-module of all morphisms from 1_A to 1_B in Idm-Iso(R). Therefore the functor $F:M(R)\longrightarrow$Idm-Iso(R) defined by $F(A)=1_A$ for $A\in M(R)$ is an isomorphism. Thus Idm-Iso(R) has enough injectives.

NOTE: It is well known that $M(R)$has a generator R. But 1_R is not a generator of End(R) and Idm(R). If $f:\alpha\longrightarrow\beta$ $(\neq 0)$ is a morphism in End(R) or Idm(R) wé can not insure existence of a morphism $g:1_R\longrightarrow\alpha$ such that $fg\neq 0$ and $\alpha g=g$. Even if End(R) and Idm(R) are C_3-categories (by Proposition 5, Theoren 7 and corollary 8), maybe they do not have enough injectives.

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