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SOLVABILITY OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

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The primary aim of this paper is to survey the solvability of linear partial differential equations and exposit the main results since the theory of distributions played its role in the field of partial differential equations. Thus the existence of *fundamental solutions* of differential polynomials, the *global existence theorems* in $C^\infty(\Omega)$ and in $\mathfrak{D}'(\Omega)$ as well as the recent results on the *local solbility* of the general linear partial differential equations and the pseudo differential equations will be our theme in this paper.

In this connection it should be mentioned that the recent result of finding the neccessary and sufficient conditions on the local solbability of the linear partial differential equations with C^∞ coefficients is one of the main achievements ever since the modern theory of linear partial differential equations were developped. It offers the theoretical foundation for the further systematic development.

1. Distributions.

Let Ω be an open subset of R^n. We denote by $x=(x_1, x_2, \cdots, x_n)$ an element in R^n. The dual of R^n will be denoted by R_n. $C_0^\infty(\Omega)$ will be the space of all complex valued infinitely differentiable functions on Ω with compact supports in Ω.

Let us choose a sequence of compact subsets $K_i (i=1, 2, \cdots)$ of Ω such that for any $i=1, 2, \cdots$, K_i is contained in the interior of K_{i+1} and $\bigcup_{i=1}^{\infty} K_i=\Omega$. Then it follows that $C_0^\infty(K_i)$, the space of all C^∞ functions with supports in K_i, becomes a Frechet space under the seminorms

$$p_m(f)=\sup_{|\alpha|\leq m} \sum_{x\in K_i} |\partial_x{}^\alpha f(x)| \quad (m=0, 1, 2, \cdots).$$

Here $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is multi-index of nonnegative integers, $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$, $\partial_x{}^\alpha=\partial_1{}^{\alpha_1}\partial_2{}^{\alpha_2}\cdots\partial_n{}^{\alpha_n}$, and $\partial_j{}^{\alpha_j}=(\partial/\partial x_j)^{\alpha_j}$.
Thus we have a sequence of function spaces

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$$C_0^\infty(K_1) \subset C_0^\infty(K_2) \subset \cdots \subset C_0^\infty(K_i) \subset \cdots.$$

Note that we have $\bigcup_{i=1}^{\infty} C_0^\infty(K_i) = C_0^\infty(\Omega)$.

We equip $C_0^\infty(\Omega)$ with a locally convex vector toplogy such that a convex set V in $C_0^\infty(\Omega)$ is open if and only if $V \cap C_0^\infty(K_i)$ is open for all $i=1, 2, \cdots$.

DEFINITION 1.1. The space of all continuous linear functional on $C_0^\infty(\Omega)$ is called *the space of distributions on* Ω. The space of distributions on Ω will be denoted by $\mathfrak{D}'(\Omega)$. Elements of $\mathfrak{D}'(\Omega)$ is called distributions. The space of all distributions with compact supports will be denoted by $\varepsilon'(\Omega)$.

2. Linear partial differential operators and pseudo differential operators.

By a *linear partial differential operator order m* on Ω with C^∞ coefficients we mean a polynomial in partial derivatives on Ω and has the form

$$p(x, D) = \sum_{|\alpha| \leq m} C_\alpha(x) D_x^\alpha.$$

Here α are the multi-indices, $D_x^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ where $D_i^{\alpha_i} = (-i(\partial/\partial x_i))^{\alpha_i}$ and $C_\alpha(x)$ are complex valued C^∞ functions on Ω.

When all the coefficients are constant on Ω, we denote the operator $p(x, D)$ by $p(D)$ and call it a differential polynomial on Ω.

Let $p(x, D)$ be a liear partial differential operator. The function $p(x, \xi)$ defined on $\Omega \times R_n$, substituting ξ in D in $p(x, D)$, is called the *symbol* of $p(x, D)$. When the order of $p(x, D)$ is m, the homogeneous term of order m

$$p_m(x, D) = \sum_{|\alpha| = m} C_\alpha(x) D_x^\alpha$$

is called the *principal part* of $p(x, D)$. $p_m(x, \xi)$ is called the *principal symbol* of $p(x, D)$.

The symbols $p(x, \xi)$ and $p_m(x, \xi)$ belongs to a general class of functions which we now define.

DEFINITION 2.1. Let Ω be an open subset of R^n. Then for $m, \rho, \delta \in R$, $0 \leq \rho, \delta \leq 1$, $S_{\rho,\delta}^m(\Omega)$ is the space of $p \in C^\infty(\Omega \times R_n)$ with the property that for any compact $K \subset \Omega$, any multi-indices α and β, there exists a constant $C_{K,\alpha,\beta}$ such that

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{K,\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

for all $x \in K$ and $\xi \in R_n$.

DEFINITION 2.2. If $p \in S_{\rho,\delta}^m(\Omega)$, then the operator $p(x, D)$ defined by

$$p(x, D)f(x) = (1/(2\pi)^n)\int p(x, \xi)e^{i<x,\xi>}f(\xi)d\xi$$

where f is the Fourier transform of f is called a ***pseudo differential operator*** of order m.

The pseudo differential operator is well defined for f in $\varepsilon'(\Omega)$ and map $\varepsilon'(\Omega)$ into $\mathfrak{D}'(\Omega)$. $p(x, \xi)$ is called the *symbol* of the pseudo differential operator $p(x, D)$.

3. Global existence theorems.

Let $p(D)$ be a differential polynomial on R^n. A distribution E on R^n is called a fundamental solution of $p(D)$ if $p(D)E=\delta$.

THEOREM 3.1. (Ehrenpreis). *Every differential polynomial has a fundamental solution.*

Proof. Let ν be the smallest integer such that $4\nu \geq n+1$ and let $\Delta = -\Sigma_{j=1}^n D_j^2$. Then there is an L^2-function $F(x)$ in R^n such that $(1-\Delta)^\nu F=\delta$. Let $E(t, x) = \text{Exp}[(1/2)(t_1^2x_1^2+\cdots+t_n^2x_n^2)]$ and define the space $H_-(t)$ as the space of the measurable functions $f(x)$ such that $E^{-1}(t, x)f(x) \in L^2$. Then it follows that there is a function $F_1 \in H_-(t)$ such that $p(D)F_1=F$. Let us set $E=(1-\Delta)^\nu F_1$. We have:

$$p(D)E=(1-\Delta)^\nu p(D)F_1=(1-\Delta)^\nu F=\delta.$$

Q. E. D.

The fundamental solution E does does not grow too fast at infinity; it is a sum of derivatives of functions which grow at most as some Exp $c|x|^2$. Also E is not too singlar; it is a finite sum of derivatives of order $\leq 2\nu$ of locally L^2-functions. The importance of the fundamental solutions lies in the following corollary.

COROLLARY. *For any $v \in \varepsilon'(R^n)$ and for any differential polynomial $p(D)$, there is $u \in \mathfrak{D}'(R^n)$ such that $p(D)u=\nu$.*

Proof. Let E be a fundamental solution of $p(D)$. Let $u = \nu * E$. Then $p(D)u=p(D)(\nu*E)=\nu*[p(D)E]=\nu*\delta=\nu$.

Q. E. D.

Let $C^\infty(\Omega)$ be the space of all complex valued C^∞-functions on Ω. We equip $C^\infty(\Omega)$ with the topology of uniform convergence on every compact subset of Ω. Thus $C^\infty(\Omega)$ becomes a Frechet space.

DEFINITION 3.1. An open subset Ω of R^n is *$p(D)$-convex* if and only if for any compact subset K of Ω there is a compact subset K' of Ω such that for any $\phi\in\varepsilon'(\Omega)$

$$\text{supp } p(-D)u\in K \text{ implies supp } u\in K'$$

where supp u means the support of u.

THEOREM 3.2. (Malgrange). *$p(D)C^\infty(\Omega)=C^\infty(\Omega)$ if and only if Ω is $p(D)$-convex.*

Proof. That Ω is $p(D)$-convex is equivalent in its functional analytic version to the fact that $p(-D)$ is one-to-one from $\varepsilon'(\Omega)$ into $\varepsilon'(\Omega)$ and the image $p(-D)\varepsilon'(\Omega)$ is weakly closed in $\varepsilon'(\Omega)$. Since $\varepsilon'(\Omega)$ is the dual of $C^\infty(\Omega)$, the latter condition is equivalent to the fact that $p(D)$ is surjective. Thus $p(D)C^\infty(\Omega)=C^\infty(\Omega)$ if and only if Ω is $p(D)$-convex.

Q. E. D.

All the geometrically convex open set is $p(D)$-convex for any differential polynomial $p(D)$. We shall offer an example of an open set which is not $p(D)$-convex.

EXAMPLE. The complement of the origin in the plane R^2 is not $(\partial/\partial x_1)$-convex.

Proof. For $\varepsilon>0$, let χ_ε be the characteristic function of the set

$$\{(x_1, x_2)\in R^2\,|\,x_1{}^2+x_2{}^2\leq 1\ \ x_2\geq\varepsilon\}.$$

Then $\text{supp}(-\partial/\partial x_1)\chi_\varepsilon$ is contained in the unit circle $x_1{}^2+x_2{}^2=1$, hence in a fixed compact subset of $R^2/\{0\}$, but this is not true of supp χ_ε.

Q. E. D.

The above theorem can be generalized to the linear partial differential operator with C^∞-coefficients. We shall define the *Sobolev space* H^s (s: real number) as the space of tempered distributions in R^n, u, whose Fourier transform $\hat{u}$ is a square integrable function with respect to the measure $(1+|\xi|^2)^s\hat{u}d\xi$. H^s is a Hilbert space and its dual is canonically isomorphic to H^{-s}. The linear partial differential operator $p(x, D)$ map H^s to H^s. We shall denote its transpose as ${}^tp(x, D)$. Thus ${}^tp(x, D)$ maps H^{-s} into H^{-s}.

THEOREM 3.3. *Let $p(x, D)$ be a linear partial differential operator with C^∞ coefficients. Then $p(x, D)C^\infty(\Omega)=C^\infty(\Omega)$ if and only if the following condition is fulfilled:*

To every compacet subset K of Ω and to every real number s, there is another compact set $K'\subset\Omega$, a real number t and a constant $B>0$ such that for all $u\in\varepsilon'(\Omega)$, the property

$$ {}^t p(x, D)u \in H^s \text{ and } supp\ {}^t p(x, D)u \subset K $$

implies

$$ supp\ u \subset K' $$

$$ u \in H^t \text{ and } |u|_t \leq B |{}^t p(x, D)u|_s. $$

Here $|u|_t$ means the norm of u in H^t. The proof of this theorem is based on the fact that the given condition is equivalent to the condition that ${}^t p(x, D)$ is an injective map from $\varepsilon'(\Omega)$ into $\varepsilon'(\Omega)$ and ${}^t p(x, D)\varepsilon'(\Omega)$ is weakly closed in $\varepsilon'(\Omega)$. (cf. [9])

Next we shall consider the existence theorem in $\mathfrak{D}'(\Omega)$. Let u be a distribution on the open set $\Omega \subset R^n$. The *singular support* of u which we shall denote by *sing supp u* is the smallest closed set F such that u is a C^∞ function in $\Omega \setminus F$. We shall start from the following definition.

DEFINITION 3.2. We shall say that Ω is *strongly $p(D)$-convex* if Ω is $p(D)$-convex and if for every compact subset K of Ω there is a compact subset K' of Ω such that for every $u \in \varepsilon'(\Omega)$

sing supp $p(-D)u \subset K$ implies sing supp $u \subset K'$.

THEOREM 3.4. (Hörmander). $p(D)\mathfrak{D}'(\Omega) = \mathfrak{D}'(\Omega)$ *if and only if Ω is strongly $p(D)$-convex.*

Proof. If Ω is strongly $p(D)$-convex, then for any continuous seminorm p on $C_0^\infty(\Omega)$ there exists a continuous seminorm q on $C_0^\infty(\Omega)$ such that for any $\phi \in C_0^\infty(\Omega)$,

$$ p(\phi) \leq q(p(-D)\phi). $$

Let then T be any distribution on Ω and choose p such that

$$ |\langle T, \phi \rangle| \leq p(\phi), \quad \phi \in C_0^\infty(\Omega). $$

It follows that the the linear form

$$ p(-D)\phi \rightarrow \langle T, \phi \rangle $$

defined on $p(-D)C_0^\infty(\Omega)$ is continuous for the topology induced by $C_0^\infty(\Omega)$. Hence according to the Hahn-Banach's theorem, it can be extended to the whole of $C_0^\infty(\Omega)$ as a linear form L satisfying

$$ |L(\psi)| \leq q(\psi). $$

This shows that there is a distribution S on Ω such that $L(\psi) = \langle S, \psi \rangle$ for all $\psi \in C_0^\infty(\Omega)$. Taking $\psi = p(-D)\phi$, we have

$$ \langle T, \phi \rangle = \langle S, p(-D)\phi \rangle = \langle p(D)S, \phi \rangle, $$

i. e., $p(D)S=T$.

Conversely if $p(D)\mathfrak{D}'(\Omega)=\mathfrak{D}'(\Omega)$, then whatever be $u\in\varepsilon'(\Omega)$, the distances to the boundary of Ω from sing supp u and from sing supp $p(-D)u$ are equal. Thus it follows immediately that Ω is strongly $p(D)$-convex, since $p(D)\mathfrak{D}'(\Omega)=\mathfrak{D}'(\Omega)$ implies $p(D)C^\infty(\Omega)=C^\infty(\Omega)$ and hence Ω is p(D)—convex.

Q. E. D.

Geometrically convex open set is strongly $p(D)$-convex for any differential polynomial $p(D)$. We shall state the following.

OPEN QUESTION. Let $p(x, D)$ be a linear partial differential operator with C^∞ coefficient. Find the neccessary and sufficient condition on Ω for $p(x, D)$ $\mathfrak{D}'(\Omega)=\mathfrak{D}'(\Omega)$. That the strong $p(x, D)$-convexity is sufficient is known (cf. [8]) but it is conjectured that strong $p(x, D)$-convexity is too strong to be a neccessary condition.

4. Local solvability.

In this section we shall introduce the neccessary and sufficient condition for the operator $p(x, D)$ to be locally solvable. We shall start from the following definition.

DEFINITION 4.1. The linear partial differential operater $p(x, D)$ with C^∞ coefficients is said to be *locally solvable* at a point x_0 of Ω if there is an open neighborhood $U\subset\Omega$ of x_0 such that, given any function $f\in C_0^\infty(U)$, there is a distribution $u\in\mathfrak{D}'(U)$ such that $p(x, D)u=f$ in U. The operator $p(x, D)$ is said to be solvable on Ω if it is locally solvable at any point of Ω.

DEFINITION 4.2. The pseudo differential operator $p(x, D)$ is said to be locally solviable in an open set if every pont x_0 of Ω has two neighborhood $U\subset V$ such that every $f\in C_0^\infty(U)$ there is a distribution u with support in V satisfying $p(x, D)u=f$ in U.

When a pseudo differential operator $p(x, D)$ is a linear partial differential operator with C^∞ coefficients, the two definitions are equivalent. In this section we shall deal with the pseudo differential operator $p(x, D)$ on Ω which can be written as $p(x, D)=p_m+p_{m-1}$ where p_{m-1} is the pseudo differential operator of order $m-1$ with symbol $p_{m-1}(x, \xi)$ in $S_{1,0}{}^{m-1}(\Omega)$ and p_m is a pseudo differential operator of order m whose symbol $p_m(x, \xi)$ is positively homogeneous in ξ-variables and is of order m, for $|\xi|>1$,

THEOREM 4.1. (Cauchy-Kovalevska). *If a linear partial differential ope-*

rator $p(x, D)$ on Ω has analytic coefficients in Ω, then for any analytic function f on Ω and for any x in Ω, there is an analytic function u in the neighborhood U of x such that $p(x, D)u=f$ on U.

The theorem 4. 1. can be proved by the method of power series expansion. It had been conjectured that C^∞ analogue of the previous theorem might be true, i. e., every $p(x, D)$ might be locally solvable. However, H. Lewy gave a counter example (cf. [7]) showing that the linear operator

$$p(x, D)=-iD_1+D_2-2(x_1+ix_2)D_3$$

is not locally solvable at any point of R^3. To introduce the criteria for local solvability we need some definitions.

DEFINITION 4. 3. Let $p(x, D)$ be a pseudo differential operator. By the *characteristic set* of $p(x, D)$ we mean $\{(x, \xi)\in\Omega\times R_n | p_m(x, \xi)=0\}$.

DEFINITION 4. 4. A pseudo differential operator $p(x, D)$ is of *principal type* if $p_m(x, \xi)=0$ and $\xi\neq 0$ implies $\partial_\xi p_m(x, \xi)\neq 0$.

In the sequel, we shall consider the operator $p(x, D)$ of the principal type only. Let $p_m(x, \xi)$ be the principal symbol of $p(x, D)$. We shall set $p_m(x, \xi)=a(x, \xi)+ib(x, \xi)$ where a and b are real and imagjnary part of $p_m(x, \xi)$, respectively. For any (x_0, ξ^0) in the characteristic set of $p(x, \xi)$ we consider the *oriented* integral curve of the *Hamilton-Jacobi equation*

$$dx/dt=\text{grad}_\xi\ a(x, \xi)$$

$$d\xi/dt=-\text{grad}_x a(x, \xi)$$

passing (x_0, ξ^0). This integral curve is called the *null-bicharacteristic* of $a(x, \xi)$.

THEOREM 4. 2. (Nirenberg-Treves-Beals-Fefferman) *Let $p(x, D)$ be a linear partial differential operator defined on $\Omega\subset R^n$ with $a(x, D)$ of the principal type. $p(x, D)$ is locally solvable on Ω if and only if, for any x_0 in Ω and for any $\xi^0\neq 0$ with (x_0, ξ^0) in the characteristic set, $b(x, \xi)$ does not change sign at (x_0, ξ^0) along the null bicharacteristic of $a(x, \xi)$ through that pont.*

Proof. We shall prove the sufficient part only. For the neccessary part confer [5].

Using the implicit function theorem, we may assume $p_m(x, D)=D_1-\lambda_1(x: D_2, D_3, \cdots, D_n)$. We may also assume that x_0 is the origin. The condition stated in the theorem is equivalent to the fact that

$$|u|_{m-1}\leq C_\varepsilon |p_m u|_0 \text{ for } u\in C_0^\infty(\Omega_\varepsilon)$$

where $\Omega_\varepsilon=\{x\in R^n \mid |x|<\varepsilon\}$ and $|u|_s$ is the Sobolev norm in H^s. The above inequality implies that, for sufficiently small $\varepsilon>0$, $pu=f$ in Ω_ε has a solution $u\in L^2(\Omega_\varepsilon)$. Thus $p=p(x, D)$ is locally solvable.

Q. E. D.

We shall state the generalization to the pseudo differential operator of the above theorem without proof (cf. [5], [6])

THEOREM 4. 3. *Let $p(x, D)$ be a pseudo differential operator defined on $\Omega\subset R^n$ with $a(x, D)$ of the principal type. Let $p(x, D)$ be locally solvable on Ω. Then for any $x_0\in\Omega$ and for any $\xi^0\neq 0$ with (x_0, ξ^0) in the characteristic set, if the function $b(x, \xi)$ on the oriented null bicharacteristic in $\Omega\times R_n$ of $a(x, \xi)$ is negative at a point, then it remains nonpositive from then along the curve.*

THEOREM 4. 4. *If the principal symbol $P_m(x, \xi)$ is analytic function on $\Omega\times R_n$, then the converse of the previous theorem is also true.*

We shall prove that the *Lewy equation* mentioned above is not locally solvable, for example, at the origin of R^3.

EXAMPLE. Lewy equation

$$p(x, D)=-iD_1+D_2-2(x_1+ix_2)D_3$$

is not locally solvable at the origin.

Proof. Let $x_0=(0,0,0)$, $\xi^0=(0,0,1)$. Then (x_0, ξ^0) belongs to the characteristic set. The null bicharacteristic of $\xi^2-2x_1\xi^3=a(x, \xi)$ passing (x_0, ξ^0) is $x_1=0$, $x_2=t$, $x_3=0$, $\xi^1=2t$, $\xi^2=0$, $\xi^3=1$. Along the null bicharacteristic

$$b(x, \xi)=-\xi^1-2x_2\xi^3=-2t-2t=-4t.$$

Thus $b(x, \xi)$ changes sign at (x_0, ξ^0) where $t=0$.

Q. E. D.

We shall conclude this paper with the following

OPEN QUESTION. Find the sufficient condition for the local solvability of the pseudo differential operator (cf. [1]).

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