Bull. Korean Math. Soc.
Vol. 15, No. 1, 1978.

A NOTE ON ALGEBRAIC K-THEORY

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1. Introduction.

The purpose of this note is to introduce the concept of algebraic K-theory for the mathematics circles in Korea.

To dispel some mystery, let us first explain the term "K-theory" [4]. Generally speaking, K-theory refers to the study of certain groups which happen to be called K_i. Topological K-theory refers to the study of these K_i-groups defined for topological spaces, and algebraic K-theory refers to the study of similar K_i-groups defined for associative rings. There is, however, one interesting historical difference between the 'topological' and the 'algebraic'. In topology the groups K_i came into existence all at once [1]. They found immediate applications, and their important role was quickly established. In algebra, however, the groups K_0 and K_1 came into existence first [2], only to be followed by an experimental period during which people tried to search for the right definition of the higher K_i's, in vain. It is only in the last few years that these higher algebraic K_i's were finally discovered; their existence and applicability have now justified the name of the subject algebraic K-theory.

Section 1 deals with K_0, Section 2 K_1, Section 3 K_2 and Section 4 the higher K_i, $i \geq 3$.

2. K_0.

We shall begin with the introduction of so-called "Grothendieck construction". For concreteness, let us first present htis basic construction in the setting of rings and modules. Let R be an arbitrary ring, and $\mu(R)$ be the family of all left R-modules. Roughly speaking, the Grothedieck construction is a process which, to any subfamily $\mathscr{C}$ in $\mu(R)$, assigns a certain abelian group, denoted by $K_0\mathscr{C}$. In detail, the construction works as follows.

For a left R-module M in the given family $\mathscr{C}$, let (M) denote the isomorphism class of M. Let F be the free abelian group on the basis $\{(M): M \in \mathscr{C}\}$, and let H be the subgroup of F generated by expressions $(M_2) -$

Work supported by the SNU-AID Project, 1976/7

$(M_1)-(M_3)$, where

$$0\to M_1\to M_2\to M_3\to 0$$

ranges over all exact sequences in $\mathscr{C}$. We then define

$$K_0\mathscr{C}=F/H \text{ (The Grothendieck group of } \mathscr{C}).$$

For $M\in\mathscr{C}$, the image of (M) in K_0 will be denoted by $[M]$. Thus, whenever we have such an exact sequence in $\mathscr{C}$, there results an equation $[M_2]=[M_1]+[M_3]$ in $K_0\mathscr{C}$. To simplify the language, it is usually permissible to say that $K_0\mathscr{C}$ is generated by the symbols $[M]$ $(M\in\mathscr{C})$, with relations given by $[M_2]=[M_1]+[M_3]$ where $M_i\in\mathscr{C}$ are as in the above exact sequence.

Like many other objects defined in mathematics, the group $K_0\mathscr{C}$ satisfies a certain universal property, which is easily described as follows. Let χ be a mapping from $\mathscr{C}$ to some abelian group $(G, +)$ such that

(1) For $M\in\mathscr{C}$, the image $\chi(M)$ depends only on the isomorphism class of M;

(2) χ is additive over short exact sequences, i.e., for each exact sequence

$$0\to M_1\to M_2\to M_3\to 0 \text{ in } \mathscr{C},$$

we have $\chi(M_2)=\chi(M_1)+\chi(M_3)$.

Then, there exists a unique group homomorphism $\chi: K_0\mathscr{C}\to G$ such that $\chi(M)=\chi([M])$ for all $M\in\mathscr{C}$.

In the definition of $K_0\mathscr{C}$, we have not imposed any substantial restriction on the subfamily $\mathscr{C}\subset\mu(R)$. In practice, however, it is desirable to require that $\mathscr{C}$ satisfy certain mild conditions, such as

(1) $\mathscr{C}$ is closed under finite direct sums;

(2) If $0\to X\to Y\to Z\to 0$ is an exact sequence in $\mu(R)$, then, $Y, Z\in\mathscr{C}\Rightarrow X\in\mathscr{C}$.

If $\mathscr{C}$ satisfies both (1) and (2), we shall say that $\mathscr{C}$ is an admissible subfamily of $\mu(R)$.

To illustrate the structure of $K_0\mathscr{C}$, let us examine a few elementary examples.

EXAMPLE 1. The simplest case is given by letting R be a field, and $\mathscr{C}$ be the (addmissible) family of all finite dimensional vector spaces over R. The mapping χ from $\mathscr{C}$ to Z given by 'dimension' is clearly additive over short exact sequences in $\mathscr{C}$. Hence it induces $\chi: K_0\mathscr{C}\to Z$ such that $\chi([M])=\dim M$. This homomorphism χ is surjective since $\chi([R])=1$. It is also injective, since $\chi([M]-[N])=0\Rightarrow\dim M=\dim N\Rightarrow M\simeq N\Rightarrow[M]-[N]=0$. Thus,

$K_0\mathscr{C} \simeq Z$ by the dimension map.

EXAMTLE 2. A similar example is given by letting R be the ring of integers and $\mathscr{C}$ be the (admissible) family of all finite abelian groups. The mapping χ from $\mathscr{C}$ to the multiplicative group of positive rationals Q^+ given by cardinality is clearly multiplicative over short exact sequences in $\mathscr{C}$. Hence it induces $\tilde{\chi} : K_0\mathscr{C} \to Q^+$ such that $\tilde{\chi}([M]) = |M|$. This homomorphism $\tilde{\chi}$ is surjective, since $\tilde{\chi}([Z/pZ]) = p$ and Q^+ is generated by the positive prime numbers. Furthermore, $\tilde{\chi}$ is injective, since $\tilde{\chi}([M] - [N]) = 1 \Rightarrow |M| = |N| \Rightarrow M$ and N have the same composition factors C_i (by 'Jordan-Hölder') $[M] = \Sigma[C_i] = [N] \in K_0\mathscr{C}$. Thus, $K_0\mathscr{C} \simeq Q^+$ by the cardinality map, and it follows that $K_0\mathscr{C}$ is a free abelian group on the basis $\{[Z/pZ] : p = \text{prime}\}$.

EXAMPLE 3. Again with $R = Z$, let $\mathscr{C}'$ be the (admissible) family of all finitely generated abelian group, which contains the family $\mathscr{C}$ in Example 2. The structure of $K_0\mathscr{C}'$, however, turns out to be completely different from $K_0\mathscr{C}$. If $0 \neq n \in Z$, we have the following exact sequence in $\mathscr{C}'$:

$$0 \to Z \xrightarrow{i} Z \to Z/nZ \to 0, \quad i(x) = nx.$$

Thus, $[Z/nZ]_{\mathscr{C}'} = [Z] - [Z] = 0$ in $K_0\mathscr{C}'$ (in contrast to $[Z/nZ]_{\mathscr{C}} \neq 0$ in $K_0\mathscr{C}$ for $n > 1$). It follows (from the Fundamental Theorem of Abelian Groups) that $K_0\mathscr{C}'$ is a cyclic group generated by the single element $[Z]$. Now, by the universal property, $M \in \mathscr{C}' \to \text{rank } M \in Z$ defines a homomorphism rank: $K_0\mathscr{C}' \to Z$. Since rank $Z = 1$, we conclude that $K_0\mathscr{C}' \simeq Z$ by rank.

Given a ring R, we shall now apply the K_0 construction to a specific subcategory $\mathscr{C}$ of $\mu(R)$. In fact, we take $\mathscr{C}$ to be $\mathscr{P}(R)$ the family of all projective, finitely generated (left) R-modules. The Grothedieck group $K_0\mathscr{P}(R)$ will be denoted simply by $K_0(R)$.

Recall that a (left) R-module P is called projective if it is a direct summand of some free R-module. Another characterization of P being projective is that any short exact sequence $0 \to X \to Y \to P \to 0$ in $\mu(R)$ must split. The latter chatacterization has the following important consequences:

(1) If a projective P is finitely generated, then P is in fact a direct summand of some finitely generated free R-module.

(2) $\mathscr{P}(R)$ is an admissible subcategory of $\mu(R)$.

(3) K_0 is functor on the category of rings. The meaning of all this jargon is actually more concrete than it sounds. Namely, it just refers to the property that, if we have a ring homomorphism $f : R \to S$, then we also have, in a natural way, an induced group homomorphism $f_* : K_0(R) \to K_0(S)$. To construct f_*, consider the mapping χ from $\mathscr{P}(R)$ to $K_0(S)$ sending $P \in \mathscr{P}(R)$ to $[S \otimes_R P] \in K_0(S)$. If $0 \to P_1 \to P_2 \to P_3 \to 0$ is an exact sequence in $\mathscr{P}(R)$,

then it splits, and so $0 \to S\otimes_R P_1 \to S\otimes_R P_2 \to S\otimes_R P_3 \to 0$ is also (split) exact. It follows that $\chi(P_2)=\chi(P_1)+\chi(P_3)$, so, by the universal property, χ defines a unique homomorphism f_*, such that $f_*([P])=[S\otimes_R P]$. Note that we have

$$(f\circ g)_*=f_*\circ g_*, \text{ and } (1_R)_*=1_{K_0(R)}.$$

The study of projective modules is not only a crucial topic in the modern chapter of ring theory, it has also found important applications to representation theory of groups and algebras, algebraic geometry, and topology. Unfortunately, given an arbitrary ring R, there is no known algorithm whereby to calculate its projective modules. On the other hand, it turns out that the abelian group $K_0(R)$ is somewhat more amenable to computations. Indeed, recent advances in algebraic K-theory have yielded various efficient means and techniques which help calculate $K_0(R)$.

3. K_1.

We shall now begin a new line of investigation by introducing the "next" functor K_1. The simplest possible definition of K_1 is given in terms of matrices, and, in this form, it was first conceived by the late topologist J. H. C. Whitehead. In [7], one of the pioneering papers in geometric topology, Whitehead noticed that, if π is the fundamental group of some space X, and if $R=Z\pi$ (the integral group ring of π) the elementary row and column transformations for matrices over the ring R have certain natural topological interpretations. Prompted by this, Whitehead introduced a certain abelian group, $Wh(\pi)$ (the Whitehead group of π), to study homotopies between spaces. For every homotopy equivalence $f: X\to Y$, Whitehead defined an invariant $\tau(f)\in Wh(\pi)$, which is such that $\tau(f)=0$ if and only if f is a simple homotopy equivalence. In topology, this invariant $\tau(f)$ has come to be known as the Whitehead torsion of f.

If one examines carefully Whitehead's definition of $Wh(\pi)$ via elementary row and column operations on matrices, on will easily see that most of the steps taken depend only on the ring $R=Z\pi$, and not on the group π. Thus, repeating these stepts for an arbitrary ring R, free from the topological context. This group is precisely the $K_1(R)$, which we shall now describe in detail.

Let $GL_n(R)$ denote the group of invertible $n\times n$ matrices over an arbitrary ring R. For $i\neq j$, and $a\in R$, let $e^a{}_{ij}$ be the elementary matrix with 1's down the diagonal, a at the ij-entry, and zeros elsewhere. We have $e^a{}_{ij}\in GL_n(R)$, since $(e^a{}_{ij})^{-1}=e^{-a}{}_{ij}$. By easy inspection, we see that left and right multiplications by $e^a{}_{ij}$ correspond to elementary row and column operations on matrices. Let $E_n(R)$ denote the subgroup of $GL_n(R)$ generated by $e^a{}_{ij}$, $a\in R$, $1\leq i\neq j\leq n$. Note that, if τ is any rectangular matrix, then any $n\times n$ matrix in block form $\sigma=\begin{pmatrix}1 & \tau\\ 0 & 1\end{pmatrix}$ is automatically in $E_n(R)$. It was further discovered by Whitehead that, if n is even, $E_n(R)$ contains another interesting class of matrices. This important discovery is known nowaday as

WHITEHEAD'S LEMMA. *If* $\alpha\in GL_m(R)$, *then*

$$\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\in E_{2m}(R).$$

If, also, $\beta\in GL_m(R)$, *then*

$$\begin{pmatrix}[\alpha,\beta] & 0\\ 0 & 1_m\end{pmatrix}\in E_{2m}(R)$$

where $[\alpha,\beta]=\alpha^{-1}\beta^{-1}\alpha\beta$ *is the commutator of* α *and* β.

Proof. The first assertion is proved by finding a sequence of elementary block' transformations which brings $\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$ to the identity. For example, one such sequence of row transformations is given by

$$\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & -\alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \alpha^{-1} & 1\end{pmatrix}\begin{pmatrix}1 & -\alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

The second assertion follows from the first, with the opportune observation that

$$\begin{pmatrix}[\alpha,\beta] & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\alpha^{-1}\beta^{-1} & 0\\ 0 & \beta\alpha\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}$$

Let us now think of $GL_n(R)$ as a subgroup of $GL_{n+1}(R)$ by identifying any $\sigma\in GL_n(R)$ with

$$\begin{pmatrix}\sigma & 0\\ 0 & 1\end{pmatrix}\in GL_{n+1}(R).$$

In this way, it is meaningful to talk about the asscending union $GL(R)=\bigcup_{n=1}^{\infty} GL_n(R)$, called the infinite general linear group over R. Since clearly $E_n(R)\subset E_{n+1}(R)$ under the avobe identification, we may also from $E(R)=\bigcup E_n(R)$. The Whitehead Lemma may now be rehashed as follows.

COROLLARY. $E(R)=[E(R),\ E(R)]=[GL(R),\ GL(R)]$.

Proof. A short commutator calculation shows that $e^a{}_{ij}$equals $[e^a{}_{ik}, e^1{}_{kj}]$ if i, j, k are distinct. Thus, we have $E_n(R)=[E_n(R), E_n(R)]$ whenever $n\geq 3$.

Whitehead's Lemma says $[GL_m(R), GL_m(R)] \subset E_{2m}(R)$ which then completes the proof.

One consequence of the Corollary is, of course, that $E(R)$ is a normal subgroup of $GL(R)$. We now define $K_1(R)$ to be the quotient group $GL(R)/E(R)$, which, in view of the Corollary, is just the abelianization of $GL(R)$. Whitehead himself, however, did not use an explicit notation for the $K_1(R)$ just defined. In his topological investigations, where $R = Z\pi$, the group $Wh(\pi)$ he introduced is not our $K_1(Z\pi)$, but is, rather, a quotient $K_1(Z\pi)$, modulo the subgroup formed by classes of the 1×1 matrices $\{(\pm g) : g \in \pi\}$.

Note that, as is the case with K_0, our new K_1 is also a functor on the category of rings. A ring homomorphism $f: R \to S$ induces a group homomorphism $GL(R) \to GL(S)$, which obviously sends $E(R)$ into $E(S)$. Hence, f induces a group homomorphism $f_*: K_1(R) \to K_1(S)$. Again, we have the functorial properties

$$(f \circ g)_* = f_* \circ g_*, \text{ and } (1_R)_* = 1_{K_1(R)}.$$

4. K_2.

Once the ground work was laid, the study of K_0 and K_1 began to draw attention from researchers in many other field, such as ring theory, number theory, group representations, quadratic froms, algebraic geometry, and, last but not least, topology. Unfailingly, these patrons bring along with them their bag of favorite things. While looking for applications of K-theory in their own subject, they make diverse contributions to the theory itself. Their concerted effort, in particular, rejuvenated the long-time speculation that there ought to exist a coherent theory of higher algebraic K_i's. The first breakthrough came in 1967, when Milnor introduced the algebraic K_2, based on earlier ideas of Steinberg. This K_2 was quickly embraced by topologists and number theorists, who were the first to discover its applications. The depth and beauty of, Milnor's K_2 made the existence of the higher K_i's an even more desirable goal [4].

To give a definition of K_2, we shall begin with relation between elementary matrices over a ring R.

Let $e^a_{ij} \in GL_n(R)$ denote the elementary matrix with entry a in the (i, j)-th place.

Following R. Steinberg [6], we introduce an abstract group defined by generators and relations which are designed to imitate the behavior of elementary matrices. Again let i, j range over all pairs of distinct integers between 1 and n and let a and b range over R.

DEFINITION. For $n \geq 3$ the Steinberg group $St_n(R)$ is the group defined by generators $x^a{}_{ij}$ subject to the relations

(1) $x^a{}_{ij} x^b{}_{ij} = x^{a+b}{}_{ij}$

(2) $[x^a{}_{ij}, x^b{}_{jl}] = x^{ab}{}_{il}$ for $i \neq l$, and

(3) $[x^a{}_{ij}, x^b{}_{kl}] = 1$ for $j \neq k$, $i \neq l$.

(In other words $St_n(R)$ is defined as a quotient $\mathcal{F}/\mathcal{R}$ where $\mathcal{F}$ denotes the free group generated by the symbols $x^a{}_{ij}$ and $\mathcal{R}$ denotes the smallest normal subgroup modulo which the above relations are valid.)

The restriction $n \geq 3$ is needed since these relations are completely inadequate when $n=2$.

Difine the canonical homomorphism

$$\phi : St_n(R) \rightarrow GL_n(R)$$

by the formular $\phi(x^a{}_{ij}) = e^a{}_{ij}$. This assignment does give rise to a homomor phism since each of the defining relations between generators of $St_n(R)$ maps into a valid identity between elementary matrices. The image $\phi(St_n(R)) \subset GL_n(R)$ is of course equal to the subgroup $E_n(R)$ generated by all elementary matrices.

Now pass to the direct limit as $n \rightarrow \infty$, thus obtaining corresponding groups and a corresponding homomorphism

$$\phi : St(R) \rightarrow GL(R).$$

Note that the image $\phi(St(R)) = E(R)$ is equal to the commutator subgroup of $GL(R)$.

DEFINITION. The kernel of the homomorphism $\phi : St(R) \rightarrow GL(R)$ will be called $K_2(R)$.

THEOREM. *The group $K_2(R)$ is the center of the Steinberg group $St(R)$.*

Thus $K_2(R)$ is an abelian group which fits into the exact sequence

$$1 \rightarrow K_2(R) \rightarrow St(R) \rightarrow GL(R) \rightarrow K_1(R) \rightarrow 1.$$

Intuitively speaking we may think of $K_2(R)$ as the set of all nontrivial relations between elementary matrices, the consequences of relations (1), (2), and (3) being the trivial relations. In fact any relation

$$e^{a_1}{}_{i_1 j_1} e^{a_2}{}_{i_2 j_2} \cdots e^{a_r}{}_{i_r j_r} = 1$$

between elementary matrices gives rise to an element $e^{a_1}{}_{i_1 j_1} e^{a_2}{}_{i_2 j_2} \cdots e^{a_r}{}_{i_r j_r}$ of $K_2(R)$, and every element of $K_2(R)$ can be obtained in this way.

As an example the matrix

$$e^1{}_{12}e^{-1}{}_{21}e^1{}_{12}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in $E_2(Z)$ represents a $90°$ relation, and hence has period 4. The relation

$$(e^1{}_{12}e^{-1}{}_{21}e^1{}_{12})^4=1$$

in $E(Z)$ gives rise to an element $(e^1{}_{12}e^{-1}{}_{21}e^1{}_{12})^4$ in $K_2(Z)$. We know that the group $K_2(Z)$ is cyclic of order 2, generated by this element $(e^1{}_{12}e^{-1}{}_{21}e^1{}_{12})^4$.

Note that K_2 is a covariant functor from rings to abelian groups. In fact every ring homomorphism $R \to R'$ clearly gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 1 \to & K_2(R) & \to & St(R) & \to & GL(R) & \to & K_1(R) & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & K_2(R') & \to & St(R') & \to & GL(R') & \to & K_1(R') & \to 1 \end{array}$$

Proof of Theorem. First recall the well known fact that an $n \times n$ matrics (a_{ij}) commutes with every $n \times n$ elementary matrix $e^b{}_{kl}$ if and only if (a_{ij}) is a diagonal matrix, with $a_{11}=a_{22}=\cdots=a_{nn}$ belonging to the center of R. For if (a_{ij}) commutes with $e^1{}_{kl}$ then direct computation shows that $a_{kl}=0$ and $a_{kk}=a_{11}$.

In particular note that no element of the subgroup

$$E_{n-1}(R) \subset E_n(R),$$

other than I, belongs to the center of $E_n(R)$. Passing to the direct limit as $n \to \infty$, it follows that the limit group $E(R)$ has trivial center.

Now if c belongs to the center of $St(R)$ then $\phi(c)$ belongs to the center of $E(R)$, hence $\phi(c)=1$.

Conversely if $\phi(y)=1$ we must prove that y commutes with every generater $x^a{}_{ij}$ of the Steinberg group. Choose an integer n large enough so that y can be expressed as a work in the generators $x^a{}_{ij}$ with $i<n$ and $j<n$. Let P_n denote the subgroup of $St(R)$ generated by the elements $x^a{}_{1n}$, $x^a{}_{2n}$, $\cdots$, and $x^a{}_{n-1n}$, where a ranges over R. This group is commutative by relation (3).

5. The higher K_i, $i \geq 3$.

After the discovery of K_2 by Milnor, the search for the higher K_i's reached major proportions. Several different definitions were proposed and tried. Finally, these provisional definitions culminated in Quillen's spectacular discovery of all the correct K_i's. In particular, the subject called Algebraic K-Theory really exists.

From his work on the Adams conjecture [5], and in particular his computation of the fibre of the Adams operation $\psi^q-1 : BU \to BU$, Quillen was motivated to propose an extremely elegant definition of higher K-groups.

THEOREM. (Quillen). *If X is a based CW complex and E is a perfect normal subgroup of $\pi_1 X$, then there is a map $X \xrightarrow{f} X^+$, unique up to homotopy, such that E is the kernel of $\pi_1(f)$ and such that the homotopy fibre F of f has the same integral homology as a point.*

If one applies this to the space $B_{Gl(R)}$ and subgroup $\mathcal{E}(R)$ of $Gl(R)$, one gets a map $B_{Gl(R)} \xrightarrow{f} B_{GlR}{}^+$ where $B_{GlR}{}^+$ has the same integral homology as $B_{Gl(R)}$, and $\pi_1 B_{Gl^+(R)} = Gl(R)/\mathcal{E}(R) = K_1(R)$.

Furthermore, it is proved that $\pi_2 B_{GlR}{}^+ = K_2(R)$ [3].

Quillen defind the higher K_i as the following

DEFINITION. $K_n(R) = \pi_n B_{Gl(R)}{}^+$, $n \geq 1$.

The space $B_{Gl(R)}{}^+$ is an extraordinary space. Quillen proves that it is a homotopy commutative and associative H-space, and in fact is an infite loop space.

References

[1] M.F. Atiyah and F. Hirzeburch, *Vector bundles and homogeneous spaces*, Proc. of Symp. in Pure Math., AMS, **3** (1961), 7-38.

[2] H. Bass, *K-theory and stable algebra*, Publ. I.H.E.S., **22** (1964), 489-544.

[3] S.M. Gersten, *'Higher K-theory of Rings' in Algebraic K-Theory* I, Lecture Notes in Mathematics **341**, Springer-Verlag, 1973.

[4] T.Y. Lam and M.K. Siu, *K_0 and K_1- An Introduction to Algebraic K-theory*, Amer. Math. Month., Vol. **82**, No.4, 1975.

[5] D.G. Quillen, *The Adams conjecture*, Topology **10** (1971), 67-80.

[6] R. Steinberg, *Générateurs, relations et revetements de groupes algébriques*, Colloq. Théorie des groupes algébriques, Bruxelles 1962, 113-127.

[7] J.H.C. Whitehead, *Simple homotopy type*, Amer. J. Math., **72** (1950), 1-57.

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