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ATIYAH-SINGER INDEX THEOREM AND ITS APPLICATION

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§ 1. Introduction.

Atiyah and Singer made their index theorem in answering a problem posed by Gelfand 1960. Gelfand's problem was to express the Fredholm operator type index of an elliptic differential operator $D : C^\infty(M) \rightarrow C^\infty(M)$ in terms of topological invariants of the manifold M and a bundle (or rather an element of $K(TM)$, where TM is a cotangent bundle of M). Not only answered for this question, they developed their theorem in various ways.

In their original proof [12], they used many machineries, especially cobordism. In a series of their subsequent papers appeared in Annals of Mathematics [1-4], they developed an elegant way of presentation. And they did not need cobordism any more. After many years of endeavors, Gilkey and Atiyah, Bott, Patodi got the index theorem in more analytic way, which is called Heat equation method. The latter method is more appropriate to the application in physics which we have in our mind.

The purpose of this paper is a presentation of the index theorem via heat equation method, and an announcement of a result which says Adler anomaly in Physics is a manifestation of the index theorem. This result will appear somewhere else.

One of us (D. P. C.) would like to express deep thanks to Dr. Kwang-sup Soh for teaching him Physics involved.

§ 2. Heat equation method.

Let E, F be bundles over M, with $A : \Gamma(E) \rightarrow \Gamma(F)$ a differential operator of order $\leq m$, where $\Gamma(E)$ (or $\Gamma(F)$) is a set of all smooth cross section of E over M.

DEF: The leading symbol of A, denoted by $\sigma(A)$, is the matrix obtained by replacing differential operator $\partial^m/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ by $\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ where ξ_i represents a cotangent vector corresponding to x_i, when A is expressed as a matrix locally and a differential operator of order m.

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DEF: A is *elliptic* $\Leftrightarrow \sigma(A)$ is invertible when $\xi=(\xi_1, \cdots, \xi_n)$ is not a zero vector.

Then in the theory of elliptic operators this nonsingularity of $\sigma(A)$ is exploited to construct a parametrix for A, that is, an operator $P: \Gamma(F) \rightarrow \Gamma(E)$ such that PA and AP both differ from the identity by smoothing operators. We have the following Fredholm type theorem.

THEOREM 1. *Suppose that* $A: \Gamma(E) \rightarrow \Gamma(F)$ *is an elliptic operator over the compact manifold* M, *then both the kernel and cokernel of* A *are finite dimensional. In particular the index of* A *is well-defined by* $index(A)=dim\ ker A - dim\ coker A$.

The proof of this theorem goes as follows. First we extend $\Gamma(E)$ (and $\Gamma(F)$) into a Hilbert space called sobolev space. Then A becames exactly a Fredholm operator. And it can be shown that the index just defined is independent of extensions. Weyl's lemma says the elements in $\ker A$ or $\ker A^*$ are smooth functions.

DEF: Laplacian $\Box$ of A means the operators on $\Gamma(E)$ and $\Gamma(F)$ respectively given by

$$\Box_E=A^*A, \quad \Box_F=AA^*$$

and $\Gamma_\lambda(E)=\{s \varepsilon \Gamma(E) \mid \Box_E s=\lambda s\}$ and define $\Gamma_\lambda(F)$ similarly.

THEOREM 2. (Hodge Theorem). *For all* $\lambda \in R$, $\Gamma_\lambda(E)$ *is finite dimensional. Further* $\Gamma_\lambda(E)=0$ *except for a discrete set of nonnegative* λ*'s and this countable sequence of subspace gives an orthogonal direct sum decomposition of the Hilbet space* $L_2(E)$ *obtained from* $\Gamma(E)$ *by completion relative to* $(\ ,\ )_E$.

Thus $L_2(E)=\oplus \Gamma_\lambda(E)$ *and* $\Gamma_0(E)=ker\ A$, $\Gamma_0(F)=coker\ A$, $A: \Gamma_\lambda(E) \rightarrow \Gamma_\lambda(F)$ *is an isomorphism. Hence* $index\ (A)=dim\ \Gamma_0(E)-dim\ \Gamma_0(F)$. *Moreover for any function* $\phi(x)$ *with* $\phi(0)=1$, *we have* $index\ (A)=\sum_\lambda \phi(\lambda)\ dim \Gamma_\lambda(E) - \sum_\lambda \phi(\lambda) dim\ \Gamma_\lambda(F)$.

The last statement suggests us to consider $e^{-\Box t}$ instead of $\Box$ itself, because the corresponding $\phi_t(x)=e^{-tx}$, and moreover $e^{-\Box t}$ is a smooth operator. Hence there exists kernel $H_t(x, y)$ such that $(e^{-\Box t} s)(x)=\int H_t(x, y) s(y) |dy|$.

Then $h_t(\Box)=\sum_\lambda e^{-\lambda t} \dim \Gamma_\lambda(E)=\int_M \operatorname{trace} H_t(x, x) |dx|$.

In terms of an orthonormal base of eigenfunctions $\{\phi_n(x)\}$ of $\Box$, trace $H_t(x, x) |dx|=\sum_n e^{-\lambda_n t} |\phi_n(x)|^2 |dx|$.

If t is small enough, then it is known that trace $H_t(x, x) |dx| \sim \sum_{k \geq -n} t^{k/2m} \mu_k(x)$

$n=\dim M$ and $m=$order of A where μ_k is a measure and a local invariant of A. Actually the crux of heat equation method is to show that μ_k is a rational function of the coefficients of A and their derivatives. The last fact was observed by Atiyah and Bott [5]. But they thought the form would be too complicated to identify them as a differential form representation of topological invariants like Chern class and Todd genus.

It was Patodi [8] who observed that the possible form of μ_k is exactly the corresponding differential form. But his algebraic machinery was very complicated. Shortly afterwards Gilkey [8] showed that μ_k should be a simple differential form representation of topological invariants on a priori grounds. Still the arguments are not easy, and very combinatorial. We owe the final and crucial simplication to Atiyah-Bott-Patodi [7]. They used classical invariants theory heavily to reduce the number of possible forms of μ_k.

Before giving the main theorem, let us make several definitions.

DEF: By a *Hermitian bundle over* M, we mean a triple $\xi=(E_\xi, h_\xi, D_\xi)$ consisting of a complex vector bundle E over M, together with a Hermitian structure h_ξ on E, and a connection D_ξ on E which preserves h_ξ.

DEF: A *joint invariant* is a function ω which assigns to every Riemann structure g in M, and every Hermitian bundle ξ over M, a q-form $\omega(g,\xi) \in \Lambda^q(M)$ such that if $f: M'\to M$ is any smooth map, then $f^*\omega(g,\xi)=\omega(f^{-1}g, f^{-1}\xi)$.

DEF: A joint invariant ω is *homogeneous* of mixed weight (k, l) if $\omega(\lambda^2 g, \mu^2\xi)=\lambda^k\mu^l\omega(g,\xi)$ $\lambda, \mu>0$.

THEOREM 3. (Gilkey, Atiyah-Bott-Patodi). *A regular joint invariant* $\omega(g,\xi)$ *of mixed weight* (k, l) *vanishes identically if* $k>0$, *or* $l\neq 0$, *while if* $k=l=0$ *then* $\omega(g,\xi)$ *has values in the ring generated by the Chern forms of* ξ *and the Pontrjagin forms of* g; *i.e.*

$$w(g,\xi)=\begin{cases} 0 \text{ if } k>0 \text{ or } l\neq 0. \\ \in Pont(g)\otimes Chern(\xi) \quad k=l=0. \end{cases}$$

Let us go back to the index as defined in theorem 1. Now index $A=h_t(\square_E)-h_t(\square_F)$.

Note that this equality holds for any $t>0$. Therefore when we replace trace $H_t(x,x)|dx|$ by a series expansion $\sum_{k\geq -n} t^{k/2m}\mu_k(\square)$, the constant term $\int_M \mu_0(\square_E)-\mu_0(\square_F)$ is equal to index A.

After some calculations Seeley obtained the following theorem,

THEOREM 4. (Seeley). *The function $\mu_k(A)$ is homogeneous of weight $k/2m$ in the coefficients of A.*

Now let us apply theorem 3 and theorem 4 to get the Hirzebruch Signature theorem. M is a manifold of dimension $2l$. Consider deRham complex with $d: \Omega^i \to \Omega^{i+1}$ and $D^*: \Omega^i \to \Omega^{i-1}$ as the adjoint of d. Let $*: \Omega^p \to \Omega^{2l-p}$ be the Hodge star operation. If we define $\tau(\alpha) = i^{p(p-1)+l} {}_*\alpha$ for $\alpha \varepsilon \Omega^p$. Then $\tau^2 \equiv \mathrm{Id}$. Denoting by $\Omega_\pm$ the ± 1-eigenspaces of τ, one verifies that $d+d^*$ interchanges Ω_+ and Ω_-. Signature operator A is $d+d^*$; $\Omega_+ \to \Omega_-$. And it is an elliptic operator. It is easy to see that the corresponding analytic index is the well known Hirzebruch Signature. Hence the Hirzebruch Signature is $\int_M \mu_0(A^*A) - \mu_0(AA^*) = \int_M w$. When theorem 3 and theorem 4 is combined, we have the following proposition.

PROPOSITION: *The differential form ω as given above is a universal polynomial in the Pontrjagin forms, say $\omega = f_k(p_1, p_2, \cdots, p_k)$, where f_k is of total degree $4k$.*

The exact form of f_k can be obtained by computing enough special cases. The basic examples are manifolds of the forms $M(k_1, \cdots, k_r) = P_{2k_1} \times \cdots \times P_{2k_r}$, $\Sigma k_i = k$, where $P_{2k} = CP^{2k}$. Then we can see that the polynomial f_k should be the one obtained by Hirzebruch himself.

Since general index theorem can be reduced to the above Hirzebruch Signature type index theorem, we have shown every essential parts of the Atiyah-Singer Index Theorem.

§3. Application in Physics.

It is well known among theoretical physicists that the various anomalies in particle physics, especially Adler anomaly (see the references for the physics involved), have a very deep relation to the index theorem. Especially in terms of differential forms, they are very similar. Anomaly occurs because of singularity of Green's function $G(x, y)$ along the diagonal. Also we could see that the index (analytic) is non zero because the parametrix of the elliptic operator given is not equal to identity. By checking the definitions of Adler anomaly, we could show that by taking proper function $\varphi(x)$, not necessarily e^{-tx} as in heat equation method in §2, Adler anomaly is just another name of the index theorem. Especially we obtained.

THEOREM 5. *If the index is not zero, then, there should exist an anomal-*

ous term in the divergence of the axial currents.

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