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SOME REMARKS ON *DCS*/x SPACES

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1. Introduction.

In paper [2], N. L. Levine proved that for an invertible spaces certain local properties become global properties.

V. M. Klassen introduced *DCS* space and *DCS*/x space. *DCS*/x space has the above property. We investigate some properties in *DCS*/x spaces.

DEFINITION 1.1. [1]. A topological space X is said to have the *disappearing closed set* (*DCS*) *property* or to be a *DCS space*, if for every proper closed subset C there is a family of open sets $\{U_i\}_{i=1}^{\infty}$ such that $U_{i+1}\subseteq U_i$ and $\cap_{i=1}^{\infty}U_i=\phi$, and there is also a sequence $\{h_i\}_{i=1}^{\infty}$ of homeomorphisms on X onto X such that $h_i(C)\subseteq U_i$ for all i.

DEFINITION 1.2. [1]. A topological space X is said to have *DCS*/x property or to be a *DCS*/x space, if for every proper closed subset C which miss x there exist two sequences $\{U_i\}_{i=1}^{\infty}$ and $\{h_i\}_{i=1}^{\infty}$ satisfying the *DCS* property.

2. Main results.

LEMMA 2.1. *For every neighborhood P of x there is a sequence $\{h_i\}_{i=1}^{\infty}$ of homeomorphisms on X onto X such that $\bigcup_{i=1}^{\infty}h_i(P)=X$.*

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be a decreasing sequence of open sets in X such that $\bigcap_{i=1}^{\infty}U_i=\phi$ and $\{h_i\}_{i=1}^{\infty}$ a sequence of homeomorphisms in X such that $h_i(X-P)\subset U_i$ for each i. Then $X-\bigcup_{i=1}^{\infty}h_i(P)\subset\bigcap_{i=1}^{\infty}U_i$, so $X\subset\bigcup_{i=1}^{\infty}h_i(P)$ since $\bigcap_{i=1}^{\infty}U_i=\phi$.

THEOREM 2.2. *If P satisfies the first axiom of countability then X satisfies the first axiom of countability.*

Proof. Let $a\in X$ and U be an open neighborhood of a. Let $\{h_i\}_{i=1}^{\infty}$ be a sequence of homeomorphisms in X such that $\bigcup_{i=1}^{\infty}h_i(P)=X$. Then $a\in h_{i_0}(P)$ for some integer i_0 and thus $h_{i_0}^{-1}(a)\in P$. Let $\{U_j\}_{j=1}^{\infty}$ be a countable open base of $h_{i_0}^{-1}(a)$ in P. Then $h_{i_0}^{-1}(a)\in U_j\subset h_{i_0}^{-1}(U)\cap P$ for some integer j. Hence $\{h_{i_0}(U_j)\mid j=1,2,\cdots\}$ is a countable open base for a in X.

THEOREM 2.3. *If P satisfies the second axiom of countability, then X satisfies the second axiom of countability.*

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be a countable base in P and let $\{h_j\}_{j=1}^{\infty}$ be a sequence of homeomorphisms in X such that $\bigcup_{j=1}^{\infty} h_j(P)=X$. Then $\{h_j(U_i)\mid i,j=1,2,\cdots\}$ is a countable base in X since $\{h_j(U_i)\mid i=1,2,\cdots\}$ is a base in $h_j(P)$ for each j.

THEOREM 2.4. *If P is a Lindelöf subspace of X then X is Lindelöf.*

Proof. Let $\{U_\alpha\}$ be an open covering of X and let $\{h_i\}_{i=1}^{\infty}$ be a sequence of homeomorphisms in X such that $\bigcup_{i=1}^{\infty} h_i(P)=X$. Since P is Lindelöf, $h_i(P)$ is Lindelöf for each i. Then there is a countable open subcovering $\{U_i^j\mid j=1,2,\cdots\}$ of $\{U_\alpha\}$ such that $h_i(P)\subset\bigcup_{j=1}^{\infty} U_i^j$ for each i. Therefore $\{U_i^j\mid i,j=1,2,\cdots\}$ is a countable open subcovering of $\{U_\alpha\}$ such that $\bigcup_{i,j} U_i^j=X$.

LEMMA 2.5. *If P (or $\bar{P}$) is a connected subspace of X then P (or $\bar{P}$) is not clopen subset of X.*

Proof. Suppose P is closed subset of X. Then $h_i(P)$ is clopen subset of X for each i, where $\{h_i\}_{i=1}^{\infty}$ is a sequence of homeomorphisms in X for $X-P$. Then $X=\bigcup_{i=1}^{\infty} h_i(P)$ and X is disjoint union. Hence there exists an integer i_0 such that $x\in h_{i_0}(P)$ and $x\notin X\bigcup_{i\neq i_0} h_i(P)$. That is, $x\in X-h_i(P)$ for each $i\neq i_0$. Let $\{U_i\}_{i=1}^{\infty}$ be a decreasing sequence of open sets in X such that $\bigcap_{i=1}^{\infty} U_i=\phi$ for $X-P$. Then $x\in h_i(X-P)\subset U_i$ for each $i\neq i_0$. Since $\{U_i\}$ is decreasing sequence, $x\in\bigcap_{i=1}^{\infty} U_i$. It is contradict to $\bigcap_{i=1}^{\infty} U_i=\phi$.

From the above lemma we obtain the following theorem.

THEOREM 2.6. *If P or $\bar{P}$ is connected subspace of X then X is connected.*

Proof. Assume that X is disconnected. Then there is a nonempty proper clopen subset O of X. O is neither P nor $\bar{P}$ by Lemma 2.5. Since $X=\bigcup_{i=1}^{\infty} h_i(P)$ and $\phi\neq 0\subsetneqq X$, there exists an i_0 such that $\phi\neq h_{i_0}(P)\cap 0\subsetneqq h_{i_0}(P)$. Suppose $h_i(P)\subset h_i(P)\cap O$ for all i, then $h_i(P)\subset O$ for all i, so $X=0$. Hence

$P \cap h_{i_0}^{-1}(0)$ is nonempty proper clopen subset of P. It is contradict.

THEOREM 2.7. *If $\bar{P}$ is regular and some U_n is regular then X is regular, where $\{U_n\}$ is a sequence in X for $X-P$ by DCS/x property.*

Proof. Let $a \in X$ and C a closed in X such that $a \notin C$.

Case I; $C \subset X-P$. (i) $a \in X-P$. Since U_n is regular and $h_n(a)$, $h_n(C) \subset U_n$, there exists two disjoint neighborhoods U, V of $h_n(a)$, $h_n(C)$ in U_n respectively. Hence $h_n^{-1}(U)$ and $h_n^{-1}(V)$ are disjoint neighborhoods of a, C in X respectively. (ii) $a \in P$. Put $C_1 = C \cap \bar{P}$. If $C_1 = \phi$, then $a \in P$, $C \subset X-\bar{P}$ and P, $X-\bar{P}$ are disjoint opens in X. If $C_1 \neq \phi$, then there are two disjoint opens U', V' on $\bar{P}$ such that $a \in U'$ and $C_1 \subset V'$. Let U, V be opens in X such that $U' = U \cap \bar{P}$ and $V' = V \cap \bar{P}$. Then $a \in U \cap P$, $C \subset V \cup (X-\bar{P})$ and $U \cap P$, $V \cup (X-\bar{P})$ are disjoint opens in X.

Case II; $C \cap P \neq \phi$. (i) $a \in P$. This case is same to case I, (ii). (ii) $a \in \bar{P}-P$. Let $C_1 = C \cap \bar{P}$, $C_2 = C \cap (X-P)$. Then there are two disjoint neighborhoods U_1', V_1' of a, C in $\bar{P}$ respectively and two disjoint neighborhoods U_2, V_2 of a, C_2 in X respectively. Let U_1 and V_1 are opens in X such that $U_1' = U_1 \cap \bar{P}$ and $V_1' = V_1 \cap \bar{P}$. Then $U_1 \cap U_2$, $V_2 \cup (V_1 \cap P)$ are disjoint opens in X such that $a \in U_1 \cap U_2$ and $C \subset V_2 \cup (V_1 \cap P)$. (iii) $a \in X-\bar{P}$. Let U, V are two disjoint neighborhoods of a and C_2 in X. Then $U \cap (X-\bar{P})$ and $V \cup P$ are two disjoint open neighborhoods of a, C in X.

Case III; $C \subset P$. (i) $a \in \bar{P}$. Let U', V' be disjoint neighborhoods of a, C in $\bar{P}$ respectively and let U, V be two opens in X such that $U' = U \cap \bar{P}$ and $V' = V \cap \bar{P}$. Then U, $V \cap P$ are two disjoint opens in X such that $a \in U$ and $C \subset V \cap P$. (ii) $a \in X-\bar{P}$. $X-\bar{P}$ and P are disjoint neighborhoods of a, C in X respectively.

THEOREM 2.8. *If $\bar{P}$ is normal and some U_n is normal, then X is normal, where $\{U_n\}$ is a sequence in X for $X-P$ by DCS/x property.*

Proof. Let C_1, C_2 be disjoint closed subsets of X.

Case I; C_1, $C_2 \subset X-P$. Since $h_n(C_1)$, $h_n(C_2) \subset U_n$ and $h_n(C_1) \cap h_n(C_2) = \phi$, we can take two disjoint neighborhoods of C_1, C_2 in X.

Case II; $C_2 \cap P \neq \phi$. Let $F_1 = C_1 \cap \bar{P}$, $F_2 = C_2 \cap \bar{P}$, $G_1 = C_1 \cap (X-P)$ and $G_2 = C_2 \cap (X-P)$. Let U_1', V_1' be disjoint neighborhoods of F_1, F_2 in $\bar{P}$ and U_2, V_2 be disjoint neighborhoods of G_1, G_2 in X. (i) $C_1 \subset X-P$. $(U_1 \cap U_2) \cup (U_2 \cap (X-\bar{P}))$ and $V_2 \cup (V_1 \cap P)$ are disjoint neighborhoods of C_1, C_2 in X, where U_1, U_2 are opens in X such that $U_1' = U_1 \cap \bar{P}$ and $V_1' = V_1 \cap \bar{P}$. (ii) $C_1 \cap P \neq \phi$. Put $W_1 = U_1 \cap P$, $W_1' = V_1 \cap P$, $W_2 = U_1 \cap U_2$, $W_2' = V_1 \cap V_2$,

$W_3=U_2\cap(X-\bar{P})$ and $W_3'=V_2\cap(X-\bar{P})$. Then $C_1\subset W_1\cup W_2\cup W_3$, $C_2\subset W_1'\cup W_2'\cup W_3'$ and $W_1\cup W_2\cup W_3$, $W_1'\cup W_2'\cup W_3'$ are disjoint opens in X. (iii) $C_1\subset P$. W_1, $V_1\cup(X-P)$ are disjoint open neighborhoods of C_1, C_2 in X respectively.

Case III; $C_2\subset P$. (i) $C_1\subset P$. It is trivial.

(ii) $C_1\cap P\neq\phi$. It is same to Case II (iii).

(iii) $C_1\subset X-P$. $(U_1\cap P)\cup(X-\bar{P})$ and $V_1\cap P$ are two disjoint neighborhoods of C_1, C_2 in X.

THEOREM 2.9. *If X and Y are topological spaces with DCS/x property and DCS/y property respectively, then $X\times Y$ is $DCS/(x,y)$ space.*

Proof. Let C be a proper closed subset of $X\times Y$ such that $(x,y)\notin C$, and let $x\in P\subset X$, $y\in Q\subset Y$ be open sets in X and Y, respectively, such that $(x,y)\in P\times Q\subset(X\times Y)-C$. Let $\{U_i\}_{i=1}^{\infty}$, $\{h_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$, $\{k_i\}_{i=1}^{\infty}$ be the open sets and homeomorphisms for $X-P$ and $Y-Q$ in X and Y, respectively. We define a sequence of homeomorphisms in $X\times Y$

$$\phi_i(a,b)=\{h_i(a),\ k_i(b)\}\ \text{for each}\ (a,b)\in X\times Y,$$

and

$$\{W_i\}_{i=1}^{\infty}=\{(U_i\times Y)\cup(X\times V_i)\}_{i=1}^{\infty}.$$

Then $\{W_i\}_{i=1}^{\infty}$ is a decreasing sequence of open sets in $X\times Y$ such that $\bigcap_{i=1}^{\infty} W_i=\phi$.

Since $C\subset(X\times Y)-(P\times Q)=\{(X-P)\times Y\}\cup\{X\times(Y-Q)\}$, $\phi_i(C)\subset\{h_i(X-P)\times Y\}\cup\{X\times k_i(Y-Q)\}\subset(U_i\times Y)\cup(X\times V_i)=W_i$ for each $i=1,2,\cdots$.

Hence $X\times Y$ is $DCS/(x,y)$ space.

THEOREM 2.10. *If P is a separable subspace of X then X is separable.*

Proof. Let A be a countable dense subset of P, and let $\{h_i\}_{i=1}^{\infty}$ be a sequence of homeomorphisms in X such that $\bigcup_{j=1}^{\infty} h_j(P)=X$. Then $D=\bigcup_{i=1}^{\infty} h_i(A)$ is a countable dense subset of X since $h_i(A)$ is a countable dense subset of $h_i(P)$ for each i.

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