Bull. Korean Math. Soc.
Vol. 14, No. 2, 1978.

HOMGREN'S THEOREM AND ITS APPLICATION TO THE WAVE OPERATOR

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Introduction.

In this paper we shall generalize the classical Homgren's theorem on the uniqueness of C^m-function solution of the Cauchy problem of the m-th order with analytic coefficients. Our generalization (cf. § 2) results in the uniqueness of the solution, C^m-function of t valued in $\mathcal{D}'(\Omega_x)$, of the Cauchy problem of the m-th order with analytic coefficients. This result mainly based on the recent results of Treves [6] on the abstract version of the Homgren's theorem.

We shall also show that the above generalization can be usefully applied in the wave operator theory (cf. § 3) in deriving the supports of the fundamental solutions of the wave operator and the uniqueness of the solutions, C^2 function of t valued in $\mathcal{D}'(R_x^n)$, of the wave equation with the initial data in $\mathcal{D}'(R_x^n)$.

1. Notations and definitions.

Let R^n be the n-dimensional Euclidean space. The element in R^n will be denoted by $x=(x^1, \cdots, x^n)$. We denote by R_n, the dual of R^n and its element will be denoted by $\xi=(\xi^1, \cdots, \xi^n)$. When we want to specialize a special component, we shall sometimes use the notation (x, t) to be an element in $R^n \times R^1$ as $(x^1, \cdots, x^n, t)$ and (ξ, τ) to be an element in the dual space $R_n \times R_1$ as $(\xi^1, \cdots, \xi^n, \tau)$.

A *linear partial differential operator* in n-independent variables $x=(x^1, \cdots, x^n)$ with complex coefficients defined in an open subset Ω of R^n is a polynomial in the partial differentiations and has the form

$$P(x, D_x)=\sum_{|\alpha|\leq m} c_\alpha(x) D_x^\alpha.$$

Here α is a multi-index, that is, an n-tuples of integers $\alpha_j \geq 0$; $|\alpha|$ denotes its length $\alpha_1+\cdots+\alpha_n$. Also

$$D_x^\alpha=(D_1)^{\alpha_1}\cdots(D_n)^{\alpha_n}$$

where $D_i^{\alpha_i}=\left(-i\frac{\partial}{\partial x^i}\right)^{\alpha_i}$.

When $P(x, D_x)$ is a partial differential operator defined in Ω, then the function $P_m(x, \xi)$ on $\Omega \times R_n$ is called the *principal symbol* where $P_m(x, \xi)$ is such that

$$P_m(x, D_x) = \sum_{|\alpha|=m} c_\alpha(x) D_x^\alpha.$$

Fixing x in Ω, we consider the set of zeros of $P_m(x, \xi)$ as a function of ξ :

$$C_p(x) = \{\xi \in R_n \mid P_m(x, \xi) = 0\}.$$

DEFINITION 1.1. The set $C_p(x)$ is called the *characteristic cone* of $P(x, D_x)$ at the point $x \in \Omega$. Every covector $\xi \in C_p(x)$ different from zero is said to be *characteristic* with respect to $P(x, D_x)$ at the point x.

Let S be a C^1-hypersurface in Ω. By this we mean a subset S of Ω such that for every x_0 of Ω has an open neighborhood U_0 where there is a C^1-function $\varphi(x)$ in U_0 with the following property; gradient of φ does not vanish anywhere in U_0 and $S \cap U_0$ is exactly the set of points $x \in U_0$ such that $\varphi(x) = 0$.

DEFINITION 1.2. The hypersurface S is said to be *noncharacteristic* if any normal covector ξ to S at any point x of S is noncharacteristic.

We shall denote by Δ_x the *Laplacian operator*

$$\left(\frac{\partial}{\partial x^1}\right)^2 + \cdots\cdots + \left(\frac{\partial}{\partial x^n}\right)^2.$$

$\mathcal{D}'(\Omega)$ will be the space of all distributions in the open subset Ω of R^n. For details of unexplained definitions and notations we refer to Treves [7].

2 General version of the classical Homgren's theorem.

In this section we shall generalize the *classical Homgren's theorem*. Let Ω be an open subset of R^n and $(-T, T)$ be an open interval in R^1. Let a partial differential operator $P(y, D_y)$ $(y = (x, t))$ is of the type

$$D_t^m + \sum_{\substack{\alpha_0 + |\alpha| \le m \\ \alpha_0 < m}} c_{\alpha_0, \alpha}(x, t) D_t^{\alpha_0} D_x^\alpha \tag{2.1}$$

where $y = (x, t) \in \Omega \times (-T, T) \subset R^{n+1}$.

We assume that the coefficient $c_{\alpha_0, \alpha}(x, t)$ are analytic in $\Omega \times (-T, T)$. Then the classical Homgren's theorem states that

if u is a C^m-function of (x, t) satisfying (2.1) in $\Omega \times (-T, T)$ and

$D_t^k u=0$ for all $x \in \Omega$ and $k=0, 1, 2, \cdots m-1$, then $u=0$ in the neighborhood of $\Omega \times \{0\}$.

The classical Homgren's theorem has been investigated in many various ways. An outstanding one of them is ***the abstract version of the Homgren's theorem*** derived from the study of the uniqueness results of linear partial differential equations which follows from the ***dual form*** of the ***abstract Cauchy-Kovalevska theorem*** (cf. [6]). The abstract version of the Homgren's theorem applied in the case of a single partial differential operator of order $m>0$, $P(y, D_y)$ [$y=(y_1, \cdots, y^N)$ is the variable in an open subset $\mathcal{U}$ of R^N] can be stated as follows.

THEOREM 2. 1. *Suppose that the coefficient of the differential operator $P(y, D_y)$ are analytic in the open set $\mathcal{U}$. Let Σ be a C^1-hypersurface in $\mathcal{U}$ subdividing $\mathcal{U}$ into two parts and nowhere charateristic with respect to $P(y, D_y)$.*

Then there is an open neighborhood of Σ in $\mathcal{U}$, $\mathcal{N}$, such that every distribution u in $\mathcal{U}$, satisfying there $P(y, D_y)\ u=0$ and vanishing on one side of Σ, also vanishes in $\mathcal{N}$.

In the case when the partial differential operator $P(y, D_y)$ is of type (2. 1), we can use the theorem 2. 1 to derive a general version of the classical Homgren's theorem applied to the solution of (2. 1) which is a C^m-function of t valued in $\mathcal{D}'(\Omega)$.

THEOREM 2. 2. *Let u be a C^m-function of t valued in $\mathcal{D}'(\Omega_x)$ satisfying $P(y, D_y)u=0$ in $\Omega \times (-T, T)$ where $P(y, D_y)$ is of the type (2. 1) with analytic coefficients. If $D_t^k u=0$ when $t=0$ for all $k=0, 1, \cdots, m-1$ in Ω, then $u=0$ in some neighborhood of $\Omega \times \{0\}$.*

Proof. Consider $\tilde{u}(x, t)=H(t)u(x, t)$ where H is the Heaviside's function, equal to one on the positive half-line and to zero on the negative half-line. Since u is a C^m-function of t and $D_t^k\ u=0$ when $t=0$ for all $k=0, 1, \cdots, m-1$, we have (cf. [5])

$$D_t^k \tilde{u}=H(t)\ D_t^k u, \quad k=0, 1, \cdots m,$$

and hence $\tilde{u}$ also satisfies $P(y, D_y)u=0$. Of course, by the definition of $\tilde{u}$, $\tilde{u}=0$ for $t<0$. Since the C^1 hypersurface $t=0$ is nowhere characteristic with respect to $P(y, D_y)$, we can apply the theorem 2. 1 to complete our proof.

3. Applications to the wave operator.

In this section we shall make applications of the theorem 2. 2 to the wave operator. We recall first that one of the fundamental solutions of the wave operator

$$\frac{\partial^2 u}{\partial t^2} - \Delta_x u$$

in R^{n+1} is

$$E(x, t) = H(t)\mathscr{U}(x, t)$$

where $\mathscr{U}(x, t)$ is a distribution whose Fourier transform with respect to x-variable is $\sin(|\xi|t)/|\xi|$ and $H(t)$ is the Heaviside function. Since $H(t)=0$ when $t<0$, $E(x, t)$ has its support in the upper half space $t \geqq 0$. Let us define the *forward light cone* Γ_+ as

$$\Gamma_+ = \{(x, t) \in R^{n+1} \mid |x|^2 \geqq t^2,\ t \geqq 0\}.$$

We shall show that an application of the theorem 2.1 says that any fundamental solution of the wave operator, which has its support in the upper half space $t \geqq 0$, has its supports in Γ_+. Thus in particular, $E(x, t)$ has its support in Γ_+.

To see this, let us recall that the fundamental solutions of the wave operator is the solution of the linear partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta_x u = \delta \tag{3.1}$$

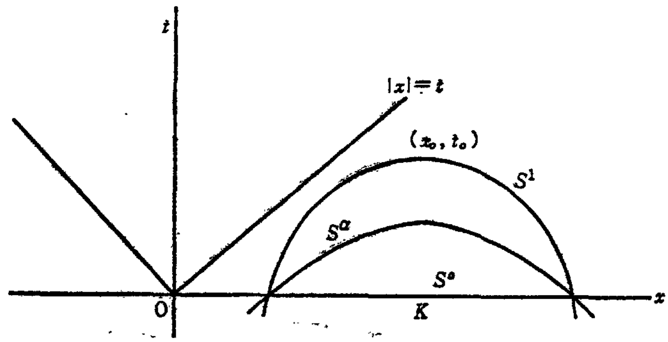
where δ is the Dirac measure at origin, it follows that if Ω is any open subset of R^n which does not contain origin, then in $\Omega \times (-T, T)$ the fundamental solutions are the solutions of the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta_x u = 0.$$

THEOREM 3.1. *Any fundamental solution of the wave operator, whose support lies in the upper half space* $t \geqq 0$, *has its support in the forward light cone* Γ_+.

Proof. It suffices to prove that the fundamental solution vanishes identically in the complement of the forward light cone Γ_+. The characteristic hypersurface of the wave operator is the hypersurface whose normal covector at every point of the hypersurface lies on the characteristic cone $|\xi|^2 = \tau^2$ where $(\xi, \tau) \in R_{n+1}$.

Let (x_0, t_0) belongs to the complement of Γ_+. Then we can find a noncharacteristic C^1-hypersurface S^1 passing (x_0, t_0) and intersecting with the noncharacteristic C^1-hypersurface $t=0$, which we shall call S^0, in the boundary of a compact set K in R_x^n such that K does not contain the origin. (Geometrically obvious construction of S^1 can be done by the C^1-partitions of unity. cf. Fig. I)



We can deform S^1 continuously to S^0 by noncharacteristic C^1-hypersurfaces $S^\alpha (0<\alpha<1)$ such that $S^\alpha \cap S^0$ equals to the boundary of K and in the upper half space $t \geqq 0$, S^β lies above S^α if $\alpha<\beta$. Since the fundamental solution vanishes on the lower half space $t<0$, by the theorem 2. 1, the fundamental solution vanishes in the neighborhood of K. As K is compact, there exists $\alpha (0<\alpha\leqq 1)$ such that the fundamental solution vanishes on the closure of the region D_α surrounded by S^α and S^0. Now if we apply the theorem 2. 1 to the noncharacteristic C^1-hypersurface S^α, we conclude that there exists $S^\beta (\alpha<\beta\leqq 1)$ such that the fundamental solution vanishes on the closure of the region D_β surrounded by S^β and S^0. Let $\gamma (0<\gamma\leqq 1)$ be the upper bound such that the fundamental solution vanishes on the closure of D_γ, surrounded by S^γ and S^0. Then it follows that $\gamma=1$, since otherwise we can apply the theorem 2. 1 on S^γ to get a contradiction. This shows that the fundamental solution vanishes at (x_0, t_0), thus completing the proof.

In the remaining of this section we shall show that the theorem 2. 2 can be applied to deduce that the Cauchy problem of the wave equation has the unique solution as C^2-function of t valued in $\mathcal{D}'(R_x^n)$. Thus we think of

(3. 2) $$\frac{\partial^2 u}{\partial t^2} - \Delta_x u = f(x, t) \text{ in } R^{n+1},$$

(3. 3) $$\text{when } t=0, \ u=u_0(x), \ \frac{\partial u}{\partial t}=u_1(x) \text{ in } R^n.$$

We shall use the following existence part of solution of (3. 2)-(3. 3) (cf. [7]):

Let u_0, u_1 be any two distributions in R^n, $f(x, t)$ any continuous function of t valued in $\mathcal{D}'(R_x^n)$. There is a C^2-function of t valued in $\mathcal{D}'(R_x^n)$ which is *a solution of* (3. 2)-(3. 3).

THEOREM 3. 2. *Let u_0, u_1 be any two distributions in R^n, $f(x, t)$ any continuous function of t valued in $\mathcal{D}'(R_x^n)$. Then the Cauchy problem* (3. 2)-(3. 3) *of the wave equation has a unique solution as a C^2-function of t valued in $\mathcal{D}'(R_x^n)$.*

Proof. Let $v(x, t)$ and $w(x, t)$ be any two solutions of (3. 2)-(3. 3) as C^2-functions of t valued in $\mathcal{D}'(R_x^n)$. We note that the characteristic cone of the wave operator is $|\xi|^2=\tau^2$. Therefore if (x_0, t_0) $(t_0>0)$ is any point in R^{n+1}, we can find a noncharacteristic C^1-hypersurface S^1 passing (x_0, t_0) and intersecting with the noncharacteristic C^1-hypersurface $t=0$, say S^0, in the boundary of a compact set K (cf.).

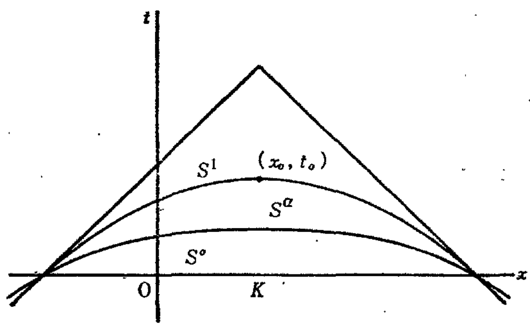


Fig. II

We can deform S^1 continuously to S^0 by the noncharacteristic C^1-hypersurface S^α $(0<\alpha<1)$ such that $S^\alpha \cap S^0$ is the boundary of K and S^β lies above S^α, if $\alpha<\beta$, in the upper half space $t\geqq 0$. Since $u=v-w$ is a C^2-function of t valued in $\mathcal{D}'(R_x^n)$ and $u=0$, $\dfrac{\partial u}{\partial t}=0$ when $t=0$, we can apply the theorem 2. 2 to conclude that u vanishes on the closure of the region D_α surrounded by S^α and S^0 for some α $(0<\alpha\leqq 1)$. Now if we apply the theorem 2. 1 to the hypersurface S^α, we see that there exists β $(\alpha<\beta\leqq 1)$ such that u vanishes on the closure of D_β, the region surrounded by S^β and S^0. Let γ $(0<\gamma\leq 1)$ be the upper bound such that u vanishes on the closure of D_γ. Then $\gamma=1$ as is shown in the proof of the theorem 2. 1. Since (x_0, t_0) $(t_0\geqq 0)$ is arbitrary and the symmetric argument can be applied for (x_0, t_0) $(t_0<0)$, this completes our proof.

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