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THE LATTICE DISTRIBUTIONS INDUCED BY THE SUM OF I. I. D. UNIFORM (0, 1) RANDOM VARIABLES

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1. Summary.

Let $X_1, X_2, \cdots, X_n$ be i. i. d. uniform (0, 1) random veriables. Let $f_n(x)$ denote the probability density function (p. d. f.) of $T_n = \sum_{i=1}^n X_i$. Consider a set $S(x\,;\,\delta)$ of lattice points defined by $S(x\,;\,\delta) = \{x \mid x = \delta + j,\ j = 0, 1, \cdots, n-1,\ 0 \leq \delta \leq 1\}$. The lattice distribution induced by the p. d. f. of T_n is defined as follow:

(1) $$f_n^{(\delta)}(x) = \begin{cases} f_n(x) & \text{if } x \in S(x\,;\,\delta) \\ 0 & \text{otherwise.} \end{cases}$$

In this paper we show that $f_n^{(\delta)}(x)$ is a probability function thus we obtain a family of lattice distributions $\{f_n^{(\delta)}(x) : 0 \leq \delta \leq 1\}$, that the mean and variance of the lattice distributions are independent of δ.

2. Main Results:

Let $f_n(x)$ be the p. d. f. of T_n, then $f_n(x)$ can be written, See Wilks [1962].

(2) $f_n(x) = (1/(n-1)!) \sum_{i=0}^n (-1)^i \binom{n}{i} (x-i)_+^{n-1}$,

where

$$x_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

First we show that $f_n^{(\delta)}(x)$ defined by (1) is probability function.

THEOREM 1. *Let $f_n^{(\delta)}(x)$ be a function defined in* (1). *Then*

$$\sum_{x \in S(x\,;\,\delta)} f_n^{(\delta)}(x) = 1.$$

Proof: Using (2), we can write

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$$\sum_{x\in S(x\,;\,\delta)} f_n^{(\delta)}(x)=(1/(n-1)!)\sum_{j=0}^{n-1}\sum_{i=1}^{j}(-1)^i\binom{n}{i}(\delta+j-i)^{n-1}$$

(3) $$=\sum_{k=0}^{n-1}\binom{n-1}{k}\delta^k\frac{1}{(n-1)!}\sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^i\binom{n}{i}(j-i)^{n-1-k}.$$

By rearranging the summation it can be shown that

(4) $$\sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^i\binom{n}{i}(j-i)^{n-1-k}=\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(n-1-j)^{n-1-k}.$$

But it is well known that the right hand side of (4) can be written as the differences of zeros, see Riordan (1958). Thus we get

$$\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(n-1-j)^{n-1-k}=\begin{cases}0 & \text{if } 0<k\le n-1\\ (n-1)! & \text{if } k=0\end{cases}$$

Hence the conclusion of theorem 1 follows.

Note that the expression (3) is a polynomial in δ of degree $(n-1)$ and the coefficients of δ^k, for $k\ge 1$, vanish.

To obtain the moments of probability function $f_n^{(\delta)}(x)$, we need the following lemma.

LEMMA: *For any positive integer m and r, we have*

(5) $$\sum_{j=0}^{m}\sum_{i=0}^{j}(-1)^i\binom{m+1}{i}(\delta+j-i)^r$$

$$=\sum_{q=0}^{r}\binom{r}{q}\delta^q\sum_{l=0}^{m}(-1)^l\binom{m}{l}(m-l)^{r-q}$$

$$=\sum_{q=0}^{r}\binom{r}{q}\delta^q\sum_{t=0}^{r-q}m!\,S(t,m),$$

where $S(t,m)$ is Stirling number of the second kind defined by

$$n!\;S(r,n)=\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-j)^r \quad \textit{or}$$

$$t^n=\sum_{r=0}^{n}t^{(r)}\;S(r,n),\quad \textit{where}\quad t^{(r)}=t(t-1)\cdots(t-r+1).$$

Now we evaluate the k-th moment of the probability function $f_n^{(\delta)}(x)$,

$$\mu_k'=(1/(n-1)!)\sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^i\binom{n}{i}(\delta+j-i)^{n-1}(\delta+j)^k$$

$$=\frac{1}{(n-1)!}\sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^i\binom{n}{i}(\delta+j-i)^{n-1}\sum_{l=0}^{k}\binom{k}{l}(\delta+j-i)^{k-l}\,i^l$$

$$=\frac{1}{(n-1)!}\sum_{l=0}^{k}\binom{k}{l}\sum_{r=0}^{l}S(l,r)\sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^i\binom{n}{i}(\delta+j-i)^{n+k-1-l}\,i^{(r)}$$

$$=\frac{1}{(n-1)!}\sum_{l=0}^{k}\binom{k}{l}\sum_{r=0}^{l}S(l,r)(-1)^r n^{(r)}\times$$

$$\sum_{q=0}^{n+k-1-l}\binom{n+k-1-l}{q}\delta^q\sum_{j=0}^{n-r-1}(-1)^j\binom{n-r-1}{j}(n-r-j)^{n+k-1-l-q}$$

$$=\frac{1}{(n-1)!}\sum_{q=0}^{n+k-1}\delta^q\sum_{l=0}^{min[n-1,n+k-1-q,k]}\binom{k}{l}\binom{n+k-1-l}{q}\times$$

$$\sum_{r=0}^{l}S(l,r)(-1)^r n^{(r)}\sum_{j=0}^{n-r-1}(-1)^j\binom{n-r-1}{j}(n-r-j)^{n+k-1-l-q}$$

$$(6)\quad=\frac{1}{(n-1)!}\sum_{q=0}^{n+k-1}\delta^q\sum_{l=0}^{min[n-1,n+k-1-q,k]}\binom{k}{l}\binom{n+k-1-l}{q}\times$$

$$\sum_{r=0}^{l}S(l,r)(-1)^r n^{(r)}\sum_{l=0}^{n+k-1-l-q}(n-r-1)!S(t,n-r-1)$$

Using (6) in conjunction with the properties of Stirling number of the second kind, the following theorem can be established.

THEOREM 2: *The mean and variance of the p.f.* $f_n^{(\delta)}(x)$ *is independent of* δ *if* $n\geq k+1$ *for* $k=1,2$. *That is,*

$$\mu=\mu_1'=n/2,\quad \mu_2'=n(3n+1)/12,\ \text{and}$$

$$\sigma^2=\mu_2'-\mu^2=n/12.$$

We note that the mean and variance of $f_n^{(\delta)}(x)$, $\delta\neq 0$, is same as the mean and variance of $f_n^{(0)}(x)$, for $n\geq 3$

However we have not obtained μ_k' for $k\geq 3$ and can not conclude whether or not they are also independent of δ. It would be interesting to find the set of values of k such that the k-th moment of $f_n^{(\delta)}(x)$ is independent of δ.

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References.

[1] Riordan, John (1958). *An Introduction to Combinatorial Analysis.* Wiley, New York.

[2] Wilks, S.S. (1962). *Mathematical Statistics*, Wiley, New York.

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