J. Korean Math. Soc.
Vol. 15, No. 1, 1978.

SOME EIGENFORMS OF THE LAPLACE-BELTRAMI OPERATORS IN A RIEMANNIAN SUBMERSION

BY YOSIO MUTŌ

Abstract. It is given in the Lecture Note [1] of Berger, Gauduchon and Mazet that, if $\pi:(\tilde{M},\tilde{g})\rightarrow(M,g)$ is a Riemannian submersion with totally geodesic fibers, $\tilde{\varDelta}$ and $\varDelta$ are Laplace operators on $(\tilde{M},\tilde{g})$ and (M,g) respectively and f is an eigenfunction of $\varDelta$, then its lift f^L is also an eigenfunction of $\tilde{\varDelta}$ with the common eigenvalue. But such a simple relation does not hold for an eigenform of the Laplace-Beltrami operator $\varDelta=d\delta+\delta d$. If ω is an eigenform of $\varDelta$ and ω^L is the horizontal lift of ω, ω^L is not in general an eigenform of the Laplace-Beltrami operator $\tilde{\varDelta}$ of $(\tilde{M},\tilde{g})$. The present author has obtained a set of formulas which gives the relation between $\tilde{\varDelta}\omega^L$ and $\varDelta\omega$ in another paper [7].

In the present paper a Sasakian submersion is treated. A Sasakian manifold $(\tilde{M},\tilde{g},\tilde{\xi})$ considered in this paper is such a one which admits a Riemannian submersion where the base manifold is a Kaehler manifold (M,g,J) and the fibers are geodesics generated by the unit Killing vector field $\tilde{\xi}$. Then the submersion is called a Sasakian submersion. If ω is an eigenform of $\varDelta$ on (M,g,J) and its lift ω^L is an eigenform of $\tilde{\varDelta}$ on $(\tilde{M},\tilde{g},\tilde{\xi})$, then ω is called an eigenform of the first kind. We define a relative eigenform of $\tilde{\varDelta}$. If the lift ω^L of an eigenform ω of $\varDelta$ is a relative eigenform of $\tilde{\varDelta}$ we call ω an eigenform of the second kind. Such objects are studied.

Introduction. All possible Riemannian submersions of the type π: $(S^N,\tilde{g}_0)\rightarrow(B,{}^Bg)$ with connected totally geodesic fibers were found by R.H. Escobales [3]. If $N=2n+1$ $(n\geqq 2)$ B can be CP^n and if $N=4n+3$ $(n\geqq 2)$ B can be QP^n. There are three special cases. According as $N=3,7,15$ we can take S^2, S^4, S^8 as B. Let us call the Riemannian metric ${}^Bg$ induced on B by such a submersion the standard Riemannian metric g_0 of the submersion. The present author took an interest in the relation which may exist between some eigenforms of the Laplace-Beltrami operator $\tilde{\varDelta}$ on $(S^N,\tilde{g}_0)$ and those of the Laplace-Beltrami operator $\varDelta$ on (B,g_0).

Received by the editors Apr. 25, 1978.

He has already studied the relation between $\tilde{\Delta}$ on $(\tilde{M}, \tilde{g})$ and Δ on $(B, {}^{B}g)$ in a Riemannian submersion $\pi: (\tilde{M}, \tilde{g}) \to (B, {}^{B}g)$ with connected totally geodesic fibers [7]. There he obtained a necessary and sufficient condition for an eigenform ω of Δ in $(B, {}^{B}g)$ to have the horizontal lift which is an eigenform of $\tilde{\Delta}$ in $(\tilde{M}, \tilde{g})$ with the same eigenvalue as ω.

But he found later that such eigenforms are scarce than he expected in some cases of $\tilde{M}=S^N$. So his interest turns to a more generous condition. This means that the lift ω^L may not be an eigenform, only it should be what we call a relative eigenform of $\tilde{\Delta}$ and the eigenvalue $\tilde{\lambda}$ of ω^L may differ from the eigenvalue λ of ω.

The present paper was first intended to treat the submersion π: $S^{2n+1} \to CP^n$. But a little more general case is studied, namely, the Sasakian submersion $\pi: (\tilde{M}, \tilde{g}, \tilde{\xi}) \to (M, g, J)$ where $(\tilde{M}, \tilde{g}, \tilde{\xi})$ is a Sasakian manifold and (M, g, J) is a Kaehler manifold.

In the present paper a lift always means a horizontal lift and is denoted $(\ )^L$. In most cases tensors and forms are denoted such as $\tilde{S}$, $\tilde{\omega}$ in $\tilde{M}$ and such as S, ω in M. A 2-form is obtained from the complex structure J and the Riemannian metric g of the Kaehler manifold. This 2-form is denoted F and its lift $\tilde{F}$.

A relative eigenform $\tilde{\omega}=\omega^L$ where ω is a form on (M, g, J) is a form satisfying

$$\pi(\tilde{\Delta}\omega^L - \tilde{\lambda}\omega^L)=0$$

where $\tilde{\lambda}$ is a number. $\tilde{\lambda}$ is called the eigenvalue of $\tilde{\omega}$.

In the present paper an eigenform always means an eigenform of Δ or $\tilde{\Delta}$. If ω is an eigenform and the lift ω^L is an eigenform, ω is called an eigenform of the first kind in the submersion. If ω is an eigenform and the lift ω^L is a relative eigenform, ω is called an eigenform of the second kind in the submersion. Any eigenform of the first kind is an eigenform of the second kind.

We define in §3 the order of a form and in §4 the coorder of a form.

Main results of the present paper are:

1°. A formula is obtained which gives the difference $\tilde{\lambda}-\lambda$ of the eigenvalue $\tilde{\lambda}$ of ω^L and the eigenvalue λ of ω when a p-form ω is an eigenform of the second kind. This difference depends on the order m of ω besides n and p (Theorem 4.2).

2°. If a p-form ω of order m is an eigenform of the second kind, then there exists a $(p-2m)$-form ω_0 of order zero satisfying

$$\omega=(F\Lambda)^m\omega_0.$$

This form ω_0 is unique and is an eigenform of the second kind. Moreover, if ω is an eigenform of the first kind, so is also ω_0 (Theorem 4.2, Theorem 4.4).

3°. If ω is an eigenform of the second kind, so is also $(F\Lambda)^k\omega_0$ (Theorem 4.4).

4°. Let ω be an eigenform of the second kind. Then $*\omega$ is also an eigenform of the second kind. The eigenvalue $\tilde{\lambda}$ of ω^L and the eigenvalue $\tilde{\lambda}_*$ of $(*\omega)^L$ satisfy $\tilde{\lambda}_* - \tilde{\lambda} = 4n-4p$ (Theorem 4.5, Corollary 4.6).

5°. Any eigenspace of Δ is spanned by some eigenforms of the second kind (Theorem 4.7).

6°. Any eigenform of the second kind of degree $p \leqq n$ is an eigenform of the first kind if it is a closed form (Theorem 5.4).

In § 1 some brief exposition of a Riemannian submersion with totally geodesic fibers is given, but a complete one is left to references. The relation between the Laplace-Beltrami operators on the total manifold and on the base manifold is given. A relative eigenform, an eigenform of the first kind and an eigenform of the second kind are defined. In § 2 fundamental formulas of a Sasakian submersion required in the sequel are given. In § 3 the order of a form is defined and some essential formulas are obtained. In § 4 the coorder of a form is defined and we obtain the Main results 1°, 2°, 3°, 4°, 5°. In § 5 some theorems about eigenforms of the first kind are obtained.

On this occasion I should like to mention the paper by A. Ikeda and Y. Taniguchi [4] whose result may have some relation with the present paper in the special case, $\pi: S^{2n+1} \to CP^n$

§ 1. Laplace-Beltrami operators of a Riemannian submersion.

We consider in this paper only Riemannian submersions $\pi: (\tilde{M}, \tilde{g}) \to (B, {}^Bg)$ with totally geodesic fibers. By such a Riemannian submersion we understand the one where fibers are complete and connected and imbedded in $\tilde{M}$ regularly as totally geodesic submanifolds. Let $\{V\}$ be any covering of $\tilde{M}$ by coordinate neighborhoods V and let x^A be the local coordinates in a neighborhood V where $A, B, \cdots = 1, \cdots, N$ and $N = \dim \tilde{M}$. Then a tensor is represented by its local components, namely the components with respect to the so-called natural frame. Let $\mathfrak{D}_F$ be the distribution parallel to the fibers. and $\mathfrak{D}_H$ the distribution normal to the fibers. Then there exists a pair of (1, 1)-tensor fields P, Q where Q projects any vector field into $\mathfrak{D}_F$ and P projects any vector field into $\mathfrak{D}_H$. They satisfy $P^2 = P$, $Q^2 = Q$, $PQ = QP = 0$, $P + Q = 1$. The local components of P, Q are denoted $P_B{}^A$, $Q_B{}^A$.

For any vector $\tilde{W} \in \tilde{M}_p$ where $p \in \tilde{M}$ we have $(P\tilde{W})^A = P_T{}^A \tilde{W}^T$, $(Q\tilde{W})^A$

$=Q_T{}^A\tilde{W}^T$ at p, and for any covector $\tilde{U}\in\tilde{M}_p{}^*$ we have $(P\tilde{U})_B=P_B{}^T\tilde{U}_T$, $(Q\tilde{U})_B=Q_B{}^T\tilde{U}_T$ at p. We also have $\tilde{W}=P\tilde{W}+Q\tilde{W}$, $\tilde{U}=P\tilde{U}+Q\tilde{U}$. Hence if we define $\tilde{W}^H$, $\tilde{W}^V$, $\tilde{U}_H$, $\tilde{U}_V$ by $P\tilde{W}=\tilde{W}^H$, $Q\tilde{W}=\tilde{W}^V$, $P\tilde{U}=\tilde{U}_H$, $Q\tilde{U}=\tilde{U}_V$, we have $\tilde{W}=\tilde{W}^H+\tilde{W}^V$, $\tilde{U}=\tilde{U}_H+\tilde{U}_V$. $\tilde{W}^H$ and $\tilde{U}_H$ are called horizontal parts while $\tilde{W}^V$ and $\tilde{U}_V$ are called vertical parts. Any of vectors and covectors is decomposed in such a way into the horizontal part and the vertical part. But in general a tensor is decomposed in a more complicated way as one can see from the concept of the tensor product. For example a (0, 2)-tensor $\tilde{S}$ with local components S_{CB} is decomposed into $\tilde{S}=\tilde{S}_{HH}+\tilde{S}_{HV}+\tilde{S}_{VH}+\tilde{S}_{VV}$. $\tilde{S}_{HH}$ is called the pure horizontal part.

The (0, 2)-tensor field and the (2, 0)-tensor field associated with the Riemannian metric $\tilde{g}$ have local components $\tilde{g}_{BA}$ and $\tilde{g}^{BA}$ respectively where $\tilde{g}_{BT}\tilde{g}^{AT}=\delta_B{}^A$. We have $P_B{}^TQ_A{}^S\tilde{g}_{TS}=0$, $P_T{}^BQ_S{}^A\tilde{g}^{TS}=0$ which means $\tilde{g}_{HV}=\tilde{g}_{VH}=0$, $\tilde{g}^{HV}=\tilde{g}^{VH}=0$. $\tilde{g}_{VV}$ represents the Riemannian metric ${}^Fg$ on the fiber and $\tilde{g}_{HH}$ the Riemannian metric ${}^Bg$ on the base manifold B. $\tilde{\nabla}$ denotes co variant differentiation in $(\tilde{M}, \tilde{g})$ and ∇ covariant differentiation in $(B, {}^Bg)$. If we write for example ∇TS where T and S are tensor fields, we mean in present paper $(\nabla T)\otimes S$.

REMARK 1. For more detailed expositions see [5], [8], [10], [11]. If $\tilde{X}^H=0$, then $\tilde{\nabla}_{\tilde{x}}P=0$, $\tilde{\nabla}_{\tilde{x}}Q=0$, $\tilde{\nabla}_{\tilde{x}}\tilde{g}_{HH}=0$.

we define a tensor field $\tilde{R}$ with local components $\tilde{R}_{CB}{}^A$ by

$$\tilde{R}_{CB}{}^A=P_C{}^TP_B{}^S(\tilde{\nabla}_TQ_S{}^A-\tilde{\nabla}_SQ_T{}^A). \tag{1. 1}$$

Then as we have $\tilde{R}_{CB}{}^TP_T{}^A=0$, we get

$$\tilde{R}=\tilde{R}_{HH}{}^V. \tag{1. 2}$$

If we use local coordinates adapted to the Riemannian submersion, we have

$$\|P_B{}^A\|=\begin{Vmatrix}\delta_i{}^h & 0\\ -\Gamma_i{}^\kappa & 0\end{Vmatrix},\quad \|Q_B{}^A\|=\begin{Vmatrix}0 & 0\\ \Gamma_i{}^\kappa & \delta_\lambda{}^\kappa\end{Vmatrix}$$

$$h, i, j, \cdots=1, \cdots, \dim B$$

$$\kappa, \lambda, \mu, \cdots=\dim B+1, \cdots, N$$

and it is easy to see that the tensor $\tilde{R}$ defined by (1. 1) is just the tensor $\tilde{R}$ which appears in [6], [7].

Let us define two (1, 3)-tensor fields $\tilde{R}_{DCB}{}^A$ and $\tilde{S}_{DCB}{}^A$ by

$$\tilde{R}_{DCB}{}^A=P_D{}^TP_C{}^SP_B{}^RQ_Q{}^A\tilde{\nabla}_T\tilde{R}_{SR}{}^Q, \tag{1. 3}$$

$$\tilde{S}_{DCB}{}^A=Q_D{}^TP_C{}^SP_B{}^RQ_Q{}^A\tilde{\nabla}_T\tilde{R}_{SR}{}^Q. \tag{1. 4}$$

If we use local coordinates adapted to the submersion again, it is easy to see that $\tilde{R}_{DCB}{}^{A}$ is just the tensor having the components $R_{kji}{}^{\kappa}$ and that $\tilde{S}_{DCB}{}^{A}$ is just the tensor having $R_{\nu ji}{}^{\kappa}$ as the leading components [6], [7].

As it seems that in studying the property of eigenforms of the second kind in a Riemannian submersion the use of local coordinates adapted to the submersion gives no special advantage, we rewrite here the formulas (3.7), (3.8), (3.9) of [7] in arbitrary local coordinates,

$$(1.5)\qquad ((\tilde{\Delta} U^L)_{H\cdots H})_{A_1\cdots A_p} = ((\Delta U)^L)_{A_1\cdots A_p} + \frac{1}{2}\sum_{1\leq a<b\leq p}\tilde{R}_{A_aA_b}{}^{Q}\tilde{R}^{TS}{}_{Q}(U^L)_{A_1\cdots T\cdots S\cdots A_p},$$

$$(1.6)\qquad ((\tilde{\Delta} U^L)_{H\cdots HV})_{A_1\cdots A_{p-1}B} = \tilde{R}^{TS}{}_{B}((\nabla U)^L)_{T,A_1\cdots A_{p-1}S} + \frac{1}{2}\sum_{a=1}^{p-1}\tilde{R}_{A_a}{}^{TS}{}_{B}(U^L)_{A_1\cdots T\cdots A_{p-1}S},$$

$$(1.7)\qquad ((\tilde{\Delta} U^L)_{H\cdots HVV})_{A_1\cdots A_{p-2}CB} = (\tilde{S}_{C}{}^{TS}{}_{B} - \tilde{R}^{PT}{}_{C}\tilde{R}_{P}{}^{S}{}_{B})(U^L)_{A_1\cdots A_{p-2}TS}.$$

Here U is a p-form in $(B, {}^{B}g)$, namely, $\omega = U_{i_1\cdots i_p}dx^{i_1}\wedge\ldots\wedge dx^{i_p}$ if we use local coordinates x^i in $(B, {}^{B}g)$, ∇U is the covariant derivative of U and $(\ )^L$ stands for the lift into $(\tilde{M}, \tilde{g})$ of a tensor or a form in $(B, {}^{B}g)$, hence $\omega^L = (U^L)_{A_1\cdots A_p}dx^{A_1}\wedge\ldots\wedge dx^{A_p}$. Indices of $\tilde{R}_{CB}{}^{A}$ are raised or lowerd with the use of $\tilde{g}^{BA}$ or $\tilde{g}_{BA}$.

REMARK 2. In (1.5) $(U^L)_{A_1\cdots T\cdots S\cdots A_p}$ stands for $(U^L)_{A_1\cdots A_{a-1}TA_{a+1}\cdots A_{b-1}SA_{b+1}\cdots A_p}$. Similar notations are used throughout the paper with such understanding.

REMARK 3. Besides (1.5), (1.6), (1.7) we have formulas such as $(\tilde{\nabla} U^L)_{H\cdots HVVV}=0, \cdots, (\tilde{\Delta} U^L)_{V\cdots V}=0$. But these are inessential in the sequel.

REMARK 4. These formulas can be obtained by straightforward calculation from the fundamental formulas of differentiation obtained by B. O'Neill. Only we have to put $T=0$ [8].

DEFINITION 1.1. If U is an eigenform of Δ and U^L is an eigenform of $\tilde{\Delta}$, U is called an eigenform of the first kind in the Riemannian submersion or shortly an eigenform of the first kind. If U is an eigenform of Δ and U^L is a relative eigenform of $\tilde{\Delta}$, namely

$$(\tilde{\Delta} U^L)_{H\cdots H}=\tilde{\lambda} U^L$$

is satisfied, U is called an eigenform of the second kind. In both cases $\tilde{\lambda}$ is called the eigenvalue of U^L.

LEMMA 1.1. *Let a p-form U be an eigenform of Δ and λ be the eigenvalue. A necessary and sufficient condition that U be an eigenform of the second kind is that there exists a number $\tilde{\lambda}$ such that*

$$(1.8)\qquad (\tilde{\lambda}-\lambda)\ (U^L)_{A_1\cdots A_p} = \frac{1}{2}\sum_{1\leq a<b\leq p} \tilde{R}_{A_a A_b}{}^{Q}\tilde{R}^{TS}{}_{Q}(U^L)_{A_1\cdots T\cdots S\cdots A_p}$$

is satisfied. A necessary and sufficient condition that U be an eigenform of the first kind is that besides (1.8) the following two equations are satisfied,

$$(1.\ 9)\qquad \tilde{R}^{TS}{}_{B}((\nabla U)^L)_{T,\, A_1\cdots A_{p-1}S} + \frac{1}{2}\sum_{a=1}^{p-1}\tilde{R}_{A_a}{}^{TS}{}_{B}(U^L)_{A_1\cdots T\cdots A_{p-1}S}=0,$$

$$(1.10)\qquad (\tilde{S}_C{}^{TS}{}_{B}-\tilde{R}^{PT}{}_{C}\tilde{R}_{P}{}^{S}{}_{B})\,(U^L)_{A_1\cdots A_{p-2}TS}=0.$$

Proof is obtained immediately from (1.5), (1.6), (1.7).

Let us define an operator $\mathfrak{R}$ which acts upon any p-form $\tilde{U}$ of $(\tilde{M}, \tilde{g})$ as follows:

$$(\mathfrak{R}\tilde{U})_{A_1\cdots A_p}\, dx^{A_1}\wedge\cdots\wedge dx^{A_p} = \frac{1}{4}\tilde{R}_{A_1A_2}{}^{Q}\tilde{R}^{TS}{}_{Q}\tilde{U}_{TSA_3\cdots A_p}dx^{A_1}\wedge\cdots\wedge dx^{A_p}.$$

Then we get from Lemma 1.1

PROPOSITION 1.2. *Let U be an eigenform of Δ. A necessary and sufficient condition that U be an eigenform of the second kind is that $\tilde{U}=U^L$ be an eigenelement of the linear operator $\mathfrak{R}$.*

The following identities are used later.

$$(1.11)\qquad \tilde{\nabla}_B(U^L)_{A_1\cdots A_p} = ((\nabla U)^L{}_{B,\,A_1\cdots A_p} - \frac{1}{2}\sum_{a=1}^{p}(\tilde{R}_{A_a}{}^{T}{}_{B}+\tilde{R}_{B}{}^{T}{}_{A_a})\,(U^L)_{A_1\cdots T\cdots A_p},$$

$$(1.12)\qquad \tilde{\nabla}_{[A_0}(U^L)_{A_1\cdots A_p]}=((\nabla U)^L)_{[A_0,\,A_1\cdots A_p]}.$$

Here and in the sequel a notation such as $[A_0\ A_1\cdots A_p]$ means that only the skew part with respect to the indices A_0, $A_1, \cdots, A_p$ is considered. Thus (1.12) is equivalent to $d(\omega^L)=(d\omega)^L$. (1.11) is a direct consequence of the fundamental formulas of differentiation in the submersion. Taking the skew part we get (1.12).

§2. A Sasakian submersion π: $(\tilde{M}, \tilde{g}, \tilde{\xi})\to(M, g, J)$.

Let $(\tilde{M}, \tilde{g}, \tilde{\xi})$ be a Sasakian manifold where dim $\tilde{M}=N=2n+1$ [2], [9]. As it is well-known this manifold is a Riemannian manifold with a unit Killing vector field $\tilde{\xi}$ and the curvature tensor $\tilde{K}_{BCD}{}^A$ satisfying

$$\tilde{K}_{TCB}{}^A\tilde{\xi}^T=\tilde{g}_{CB}\tilde{\xi}^A-\tilde{\xi}_B\delta_C{}^A \tag{2. 1}$$

where $\tilde{\xi}_B=\tilde{g}_{BT}\tilde{\xi}^T$. We consider the case where the Sasakian manifold admits a Riemannian submersion

$$\pi:(\tilde{M}, \tilde{g}, \tilde{\xi})\to(M, g, J)$$

such that $\tilde{\xi}$ is a vertical vector field and (M, g, J) is a Kaehler manifold with the complex structure J [5], [11]. This Kaehler manifold is such that dim $M=2n$ and if we take local coordinates x^h and J is represented by $F_j{}^i$ $(h, i, j, \cdots=1, \cdots, 2n)$, then the lift of the 2-form $F_{ji}=F_j{}^tg_{ti}$ is the 2-form

$$\tilde{F}_{BA}=\tilde{\nabla}_B\tilde{\xi}_A=-\tilde{\nabla}_A\tilde{\xi}_B. \tag{2. 2}$$

The projection tensors P, Q of this submersion are given by

$$P_B{}^A=\delta_B{}^A-\tilde{\xi}_B\tilde{\xi}^A,\quad Q_B{}^A=\tilde{\xi}_B\tilde{\xi}^A. \tag{2. 3}$$

We call such a Riemannian submersion a Sasakian submersion [6]. We assume $n\geqq2$.

We get from (1.1) and (2.3)

$$\tilde{R}_{CB}{}^A=(\tilde{\nabla}_C\tilde{\xi}_B-\tilde{\nabla}_B\tilde{\xi}_C)\tilde{\xi}^A=2\tilde{\nabla}_C\tilde{\xi}_B\tilde{\xi}^A=2\tilde{F}_{CB}\tilde{\xi}^A. \tag{2. 4}$$

In view of (2.1) we get

$$\tilde{\nabla}_D\tilde{R}_{CB}{}^A=2\tilde{\nabla}_D\tilde{\xi}^A\tilde{\nabla}_C\tilde{\xi}_B+2(\tilde{\xi}_C\tilde{g}_{DB}-\tilde{\xi}_B\tilde{g}_{DC})\tilde{\xi}^A$$

and as $\tilde{\xi}^A$ is a unit Killing vector

$$\tilde{R}_{DCB}{}^A=0,\quad \tilde{S}_{DCB}{}^A=0. \tag{2. 5}$$

In a Sasakian submersion (1.10) is always satisfied since the fiber is one dimensional. (1.8) and (1.9) take the form

$$(\tilde{\lambda}-\lambda)\ (U^L)_{A_1\cdots A_p} =2\textstyle\sum_{1\leq a<b\leq p}\tilde{F}_{A_aA_b}\tilde{F}^{TS}(U^L)_{A_1\cdots T\cdots S\cdots A_p}, \tag{2.6}$$

$$\tilde{F}^{TS}((\nabla U)^L)_{T,SA_2\cdots A_p}=0 \tag{2.7}$$

respectively in view of (2.4) and (2.5).

Let us recollect the following property of the contact structure

$$\tilde{F}_{BT}\tilde{F}_A{}^T=\tilde{g}_{BA}-\tilde{\xi}_B\tilde{\xi}_A,\quad \tilde{F}_{BT}\tilde{\xi}^T=0,\quad \tilde{F}^{TS}\tilde{F}_{TS}=2n, \tag{2.8}$$

$$\tilde{\nabla}_C\tilde{F}_{BA}=\tilde{\nabla}_C\tilde{\nabla}_B\tilde{\xi}_A=-\tilde{g}_{CB}\tilde{\xi}_A+\tilde{g}_{CA}\tilde{\xi}_B. \tag{2.9}$$

Now F_{ji} and $\tilde{F}_{BA}$ are 2-forms of (M,g,J) and $(\tilde{M},\tilde{g},\tilde{\xi})$ respectively which we write F and F^L. We get from (1.5), (1.6), (1.7) and (2.4), (2.5)

$$((\tilde{\Delta}\tilde{F})_{HH})_{BA}=((\Delta F)^L)_{BA}+2\tilde{F}_{BA}\tilde{F}^{TS}\tilde{F}_{TS}=((\Delta F)^L)_{BA}+4n\tilde{F}_{BA},$$

$$((\tilde{\Delta}\tilde{F})_{HV})_{BA}=2\tilde{\xi}_A\tilde{F}^{TS}((\nabla F)^L)_{T,BS},$$

$$((\tilde{\Delta}\tilde{F})_{VV})_{BA}=0.$$

But F is a harmonic form and $\nabla F=0$. Hence we get

$$\tilde{\Delta}\tilde{F}_{BA}=4n\tilde{F}_{BA} \tag{2.10}$$

which proves that F is an eigenform of the first kind with $\lambda=0$ and $\tilde{\lambda}=4n$. Thus we have

PROPOSITION 2.1. *In a Sasakian submersion the 2-form F is an eigenform of the first kind where $\lambda=0$, $\tilde{\lambda}=4n$.*

From (1.11) we get

$$\tilde{F}^{TS}\tilde{\nabla}_T(U^L)_{SA_2\cdots A_p}$$

$$=\tilde{F}^{TS}((\nabla U)^L)_{T,SA_2\cdots A_p}-\frac{1}{2}\tilde{F}^{TS}(\tilde{R}_S{}^Q{}_T+\tilde{R}_T{}^Q{}_S)(U^L)_{QA_2\cdots A_p}$$

$$-\frac{1}{2}\tilde{F}^{TS}\sum_{a=2}^{p}(\tilde{R}_{A_a}{}^Q{}_T+\tilde{R}_T{}^Q{}_{A_a})(U^L)_{SA_2\cdots Q\cdots A_p}.$$

In the second member, the second term vanishes in view of $\tilde{F}^{TS}+\tilde{F}^{ST}=0$. The third term vanishes because of (2.4) and (2.8). Hence we have

$$\tilde{F}^{TS}\tilde{\nabla}_T(U^L)_{SA_2\cdots A_p}=\tilde{F}^{TS}((\nabla U)^L)_{T,SA_2\cdots A_p}$$

which proves the

LEMMA 2.2. *The $(p-1)$-form $\tilde{F}^{TS}\tilde{\nabla}_T(U^L)_{SA_2\cdots A_p}$ is the lift of the $(p-1)$-form $F^{ts}\nabla_t U_{si_2\cdots i_p}$.*

In a Sasakian submersion (1.5), (1.6), (1.7) become

$$((\tilde{\Delta}U^L)_{H\cdots H})_{A_1\cdots A_p}=((\Delta U)^L)_{A_1\cdots A_p} \tag{2.11}$$

$$+2\sum_{1\leq a<b\leq p}\tilde{F}_{A_aA_b}\tilde{F}^{TS}(U^L)_{A_1\cdots T\cdots S\cdots A_p},$$

(2.12) $$((\tilde{\Delta}U^L)_{VH\cdots H})_{BA_2\cdots A_p}=2\tilde{\xi}_B\tilde{F}^{TS}((\nabla U)^L)_{T,SA_2\cdots A_p},$$

(2.13) $$(\tilde{\Delta}U^L)_{VVH\cdots H}=0.$$

§ 3. Order of eigenforms and eigenvalues.

First let us define two linear operators $\tilde{T}$ and $\tilde{\Xi}$ acting upon a p-form of a Sasakian manifold.

DEFINITION 3.1. When the operator $\tilde{T}$ acts upon a p-form $\tilde{V}$ with components $\tilde{V}_{A_1\cdots A_p}$ we have

(3. 1) $$(\tilde{T}\tilde{V})_{A_3\cdots A_p}=\tilde{F}^{TS}\tilde{V}_{TSA_3\cdots A_p},$$
$$\tilde{T}^k\tilde{V}=0 \quad \text{if} \quad 2k>p,$$

where $\tilde{T}^2=\tilde{T}\tilde{T}$, $\tilde{T}^{k+1}=\tilde{T}\tilde{T}^k$.

DEFINITION 3.2. When the operator $\tilde{\Xi}$ acts upon a p-form $\tilde{V}$ we have

(3. 2) $$(\tilde{\Xi}\tilde{V})_{A_1\cdots A_{p+2}}=\tilde{F}_{[A_1A_2}\tilde{V}_{A_3\cdots A_{p+2}]}$$

hence $\tilde{\Xi}\tilde{V}=\tilde{F}\wedge\tilde{V}$.

If $\tilde{*}$ stands for the star operator of $(\tilde{M}, \tilde{g}, \tilde{\xi})$, we have

$$(2n-1-p)!\,\tilde{*}\tilde{\Xi}\tilde{V}=(2n+1-p)!\,\tilde{T}\tilde{*}\tilde{V}$$

in consequence of

$$\varepsilon^{A_1\cdots A_N}\tilde{F}_{A_1A_2}\tilde{V}_{A_3\cdots A_{p+2}}=\tilde{F}_{TS}\varepsilon^{TSA_3\cdots A_N}\tilde{V}_{A_3\cdots A_{p+2}}.$$

DEFINITION 3.3. A p-form $\tilde{V}$ is said to be of order m if $\tilde{V}$ satisfies

$$\tilde{T}^m\tilde{V}\neq 0, \qquad \tilde{T}^{m+1}\tilde{V}=0.$$

DEFINITION 3.4. A p-form $\tilde{V}$ is said to be a horizontal p-form if $\tilde{V}_{TA_2\cdots A_p}\tilde{\xi}^T=0$.

As $\tilde{F}$ is a horizontal 2-form, $\tilde{T}\tilde{V}$ and $\tilde{\Xi}\tilde{V}$ are horizontal forms if $\tilde{V}$ is a horizontal form. We get

LEMMA 3.1. *If $\tilde{V}$ is a horizontal q-form, we have*

(3.3)$_1$ $$(q+2)(q+1)\tilde{T}\tilde{\Xi}\tilde{V}$$
$$=4(n-q)\tilde{V}+q(q-1)\tilde{\Xi}\tilde{T}\tilde{V},$$

(3.3)$_k$ $$(q+2k)(q+2k-1)\tilde{T}\tilde{\Xi}^k\tilde{V}$$
$$=4k(n+1-q-k)\tilde{\Xi}^{k-1}\tilde{V}+q(q-1)\tilde{\Xi}^k\tilde{T}\tilde{V}.$$

Proof. We have the following identity if $q \geqq 2$

$$(3.4)\qquad \begin{aligned}&(q+2)(q+1)(\tilde{\Xi}\tilde{V})_{A_1\cdots A_{q+2}}\\ &=2[\tilde{F}_{A_1A_2}\tilde{V}_{A_3\cdots A_{q-2}}\\ &\quad -\textstyle\sum_{a=3}^{q+2}(\tilde{F}_{A_aA_2}\tilde{V}_{A_3\cdots A_1\cdots A_{q+2}}\\ &\qquad +\tilde{F}_{A_1A_a}\tilde{V}_{A_3\cdots A_2\cdots A_{q+2}})\\ &\quad +(q(q-1)/2)\tilde{F}_{[A_3A_4}\tilde{V}_{|A_1A_2|A_5\cdots A_{q+2}]}]\end{aligned}$$

where we understand by $[A_3A_4|A_1A_2|A_5\cdots A_{q+2}]$ that we take the skew part with respect to only $A_3,\ A_4,\ A_5, \cdots, A_{q+2}$. As $\tilde{F}_{BA}$ satisfies (2.8) we immediately get $(3.3)_1$. If $q<2$ we get $(3.3)_1$ directly. Assume that we have proved $(3.3)_{k-1}$. As we have $\tilde{T}\tilde{\Xi}^k\tilde{V}=\tilde{T}\tilde{\Xi}(\tilde{\Xi}^{k-1}\tilde{V})$ where $\tilde{\Xi}^{k-1}\tilde{V}$ is a $(q+2k-2)$-form, we get

$$\begin{aligned}&(q+2k)(q+2k-1)\tilde{T}\tilde{\Xi}^k\tilde{V}\\ &\qquad =4(n-q-2k+2)\tilde{\Xi}^{k-1}\tilde{V}\\ &\qquad\quad +(q+2k-2)(q+2k-3)\tilde{\Xi}\tilde{T}\tilde{\Xi}^{k-1}\tilde{V}.\end{aligned}$$

Then applying $(3.3)_{k-1}$ to the second term of the second member, we get $(3.3)_k$.

COROLLARY 3.2. *If $\tilde{V}$ is a horizontal q-form of order zero, we have*

$$(3.5)_k\qquad (q+2k)(q+2k-1)\tilde{T}\tilde{\Xi}^k\tilde{V}=4k(n+1-q-k)\tilde{\Xi}^{k-1}\tilde{V}.$$

LEMMA 3.3. *If a horizontal q-form $\tilde{V}$ where $q=n$ satisfies $\tilde{\Xi}\tilde{V}=0$, then $\tilde{V}=0$ or $q=n+m$ where m is the order of $\tilde{V}$.*

Proof. From $(3.3)_1$ we get

$$4(n-q)\tilde{V}=-q(q-1)\tilde{\Xi}\tilde{T}\tilde{V}.$$

If $\tilde{V}\neq 0$, then $q\geqq 2m$. Applying $(3.3)_1$ again we get

$$\begin{aligned}&4(n-q)\tilde{T}\tilde{V}=-q(q-1)\tilde{T}\tilde{\Xi}(\tilde{T}\tilde{V})\\ &\qquad =-\{4(n-q+2)\tilde{T}\tilde{V}+(q-2)(q-3)\tilde{\Xi}\tilde{T}^2\tilde{V}\},\end{aligned}$$

hence

$$8(n-q+1)\tilde{T}\tilde{V}+(q-2)(q-3)\tilde{\Xi}\tilde{T}^2\tilde{V}=0.$$

Then we get

$$4(k+1)(n-q+k)\tilde{T}^k\tilde{V}+(q-2k)(q-2k-1)\tilde{\Xi}\tilde{T}^{k+1}\tilde{V}=0$$

by mathematical induction. This formula is valid until k satisfies $(q-2k)\cdot$

$(q-2k-1)=0$, hence for $k=0, 1, \cdots, m$. If we put $k=m$ we get $(n-q+m)\tilde{T}^m\tilde{V}=0$, which proves $q=n+m$.

This lemma implies that if a q-form $\tilde{V}$ satisfies $\tilde{\Xi}\tilde{V}=0$ and $q<n$, then $\tilde{V}=0$.

Lemma 3.3 does not treat the case $q=n$. But we have

LEMMA 3.4. *If a horizontal n-form* $\tilde{V}$ *satisfies* $\tilde{\Xi}\tilde{V}=0$ *and* $\tilde{V}\neq 0$, *then* $\tilde{V}$ *is of order zero.*

Proof. From $(3.3)_1$ and the assumption we get $\tilde{\Xi}\tilde{T}\tilde{V}=0$. As $\tilde{T}\tilde{V}$ is an $(n-2)$-form we get $\tilde{T}\tilde{V}=0$.

In a Kaehler manifold (M, g, J) any form has a similar property. First let us define operators T and Ξ.

DEFINITION 3.5. In a Kaehler manifold T and Ξ are defined by

$$(TV)_{i_3\cdots i_p}=F^{ts}V_{tsi_3\cdots i_p},$$

$$TV=0 \quad \text{if} \quad p<2,$$

$$(\Xi V)_{i_1\cdots i_{p+2}}=F_{[i_1 i_2}V_{i_3\cdots i_{p+2}]}$$

where V is a p-form.

DEFINITION 3.6. A p-form V of a Kaehler manifold is said to be of order m if V satifies $T^mV\neq 0$, $T^{m+1}V=0$.

If $\tilde{V}$ is a lift V^L, then $\tilde{V}$ is horizontal. As $\tilde{F}$ is a lift, we immediately obtain the

LEMMA 3.5. $\tilde{T}V^L=(TV)^L$, $\tilde{\Xi}V^L=(\Xi V)^L$. *If* V *is of order* m, *then* V^L *is of order* m.

Thus considering the case of $\tilde{V}=V^L$ we obtain from $(3.1)_1$

${}^B(3.3)_1$ $\qquad (q+2)(q+1)T\Xi V=4(n-q)V+q(q-1)\Xi TV.$

This result is so to say the projection of $(3.3)_1$. Similarly we get projections ${}^B(3.3)_k$ and ${}^B(3.5)_k$ of $(3.3)_k$ and $(3.5)_k$.

Now let us assume that a p-form $U\neq 0$ is an eigenform of the second kind in a Sasakian submersion. Then in view of (2.6) we get

$$(3.6) \qquad (\tilde{\lambda}-\lambda)U^L=p(p-1)\tilde{\Xi}\tilde{T}U^L$$

which is equivalent to

$${}^B(3.6) \qquad (\tilde{\lambda}-\lambda)U=p(p-1)\Xi TU.$$

Then by the action of T upon ${}^B(3.6)$ we get

$$\left[\frac{1}{2}(\tilde{\lambda}-\lambda)-2(n-p+2)\right]TU=\frac{1}{2}(p-2)(p-3)\Xi T^2U$$

in view of ${}^B(3.3)$. By the iterated action of T we get

$$\left[\frac{1}{2}(\tilde{\lambda}-\lambda)-2k(n-p+k+1)\right]T^kU = \frac{1}{2}(p-2k)(p-2k-1)\Xi T^{k+1}U. \tag{3.7$_k$}$$

Suppose $(\tilde{\lambda}-\lambda)/2-2k(n-p+k+1)$ does not vanish for $=0, 1, 2, \cdots$. Then there can be no number k satisfying $T^kU\neq 0$, $T^{k+1}U=0$ simultaneously, contrary to the fact that $T^{k+1}U$ must vanish for some k, hence the equation

$$x(x+1+n-p)-(\tilde{\lambda}-\lambda)/4=0 \tag{3.8}$$

has at least one solution which is a non-negative integer. Let m' be the smallest one. Then we have $m'\leqq m$ where m is the order of U. Consequently the first member of $(3.7)_k$ does not vanish for $k=0, 1, \cdots, m'-1$ and we have $p-2m'+1>0$ and $T^{m'}U\neq 0$. If we put $k=m'$ we get from (3.7)

$$(p-2m')(p-2m'-1)\Xi T^{m'+1}U=0.$$

Then we have four possible cases.

(i) If $p=2m'$ or $p=2m'+1$ we get $T^{m'+1}U=0$. From this and $T^{m'}U\neq 0$ we get $m'=m$.

If $p>2m'+1$ we get $\Xi T^{m'+1}U=0$. As the $(p-2m'-2)$-form $T^{m'+1}U$ is of order $m-m'-1$, we get from Lemma 3.3 the following three possible cases.

(ii) $T^{m'+1}U=0$,

(iii) $p-2(m'+1)=n$,

(iv) $p-2(m'+1)=n+m-(m'+1)$.

If $p-2(m'+1)=n$, we get from Lemma 3.4 $m-m'-1=0$.

(i) and (ii) give $m'=m$. From any of (iii) and (iv) we get $m+m'=p-n-1$. As (3.8) has two roots whose sum is $p-n-1$, we can conclude that $x=m$ satisfies (3.8) and we have in any case

$$\tilde{\lambda}-\lambda=4m(m+1+n-p). \tag{3.9}$$

If $m'\neq m$, we get from $0\leqq m'<m$ and $m+m'=p-n-1$

$$n+m+1\leqq p\leqq n+2m.$$

This shows that, if p and m satisfy $p<n+m+1$ or $p\geqq n+2m+1$, then we have $m'=m$.

If $p<n+m+1$ and $m'=m$, we get from ${}^B(3.6)$ and (3.7)

$$U=\frac{p!\,(m+n-p)!}{4^m m!\,(p-2m)!\,(2m+n-p)!}\Xi^m T^m U. \tag{3.10}$$

Then we see that there exists a $(p-2m)$-form U_0 such that

$$U=\Xi^m U_0, \qquad TU_0=0. \tag{3.11}$$

In the Sasakian manifold $(\tilde{M}, \tilde{g}, \tilde{\xi})$ over (M, g, J) we have

$$U^L=\tilde{\Xi}^m U_0{}^L, \qquad \tilde{T} U_0{}^L=0. \tag{3.12}$$

Thus we have obtained the following lemma.

LEMMA 3.6 *Let a p-form U of order m be an eigenform of the second kind in a Sasakian submersion. Then $\tilde{\lambda}-\lambda$ is given by* (3.9). *If $p<n+m+1$ is satisfied, then we have* (3.10), (3.11), (3.12).

REMARK. If $m=0$ (3.10) is interpreted as $U=U$.

COROLLARY 3.7. *Let a p-form U of the base manifold be an eigenform of the second kind in a Sasakian submersion. Then $\tilde{\lambda}=\lambda$ if and only if the order of U is zero or $p-n-1$.*

§ 4. Order and coorder of eigenforms.

DEFINITION 4.1. The coorder of a p-form U of a Kaehler manifold is said to be l if $\Xi^l U\neq 0$, $\Xi^{l+1}U=0$.

There always exists such a number l and satisfies $p+2l\leqq 2n$. As we have

$$(2n-p-2)!*\Xi U=(2n-p)!\,T*U,$$

we get $*\Xi^k U=\alpha_{k,p}T^k*U$ where $\alpha_{k,p}$ is some positive number. Hence the coorder l of U is the order of $*U$.

If V is a q-form we get from ${}^B(3.3)_1$

$$q(q-1)\Xi TV=(q+2)(q+1)T\Xi V-4(n-q)V. \tag{4.1}$$

If a p-form U is an eigenform of the second kind, we get from ${}^B(3.6)$ and (4.1) in which we put $V=U$ and $q=p$,

$$\{\tilde{\lambda}-\lambda+4(n-p)\}U=(p+2)(p+1)T\Xi U. \tag{4.2}$$

By iterated action of Ξ we get

$$\begin{aligned}\{\tilde{\lambda}-\lambda+4k(n-p-k+1)\}\Xi^{k-1}U \\ =(p+2k)(p+2k-1)T\Xi^k U.\end{aligned} \tag{4.3}$$

To obtain this we use (4.1) putting $V=\Xi U$, $\Xi^2 U$ and so on.

If $\tilde{\lambda}-\lambda+4k(n-p-k+1)$ does not vanish for $k=1, 2, \cdots$ there can not exist a number l such that $\Xi^l U \neq 0$, $\Xi^{l+1} U = 0$, contrary to the assumption that U is of coorder l. Hence the equation

$$x(x-1-n+p)-(\tilde{\lambda}-\lambda)/4=0 \tag{4.4}$$

has at least one natural number as solution. Let l' be the smallest one. Then from the definition of the coorder l' must satisfy $l' \leqq l+1$. If we put $k=l'$ in (4.3) we get $T\Xi^{l'} U=0$, hence $\Xi T^{l'} *U=0$. $T^{l'} *U$ is a $(2n-p-2l')$-form of order $l-l'$ as $*U$ is a $(2n-p)$-form of order l. In view of Lemma 3.3 and Lemma 3.4 we get three possible cases,

(i) $T^{l'} *U=0$,

(ii) $2n-p-2l'=n+l-l'$, namely, $l+l'=n-p$,

(iii) $2n-p-2l'=n$ and $l-l'=0$, namely, $n-p=2l=2l'$

If we have (i) we get $\Xi^{l'} U=0$, hence $l'=l+1$. Then $l+1$ is a solution of (4.4). If we have (ii) or (iii) we get $(l+1)+l'=n-p+1$. Hence $l+1$ is a solution of (4.4). In any case we have

$$\tilde{\lambda}-\lambda=4(l+1)(l-n+p).$$

As we have (3.9) already, we get

$$\begin{aligned} m(m+1+n-p) &= (l-n+p)(l+1) \\ &= (l-n+p)\{(l-n+p)+1+n-p\} \end{aligned}$$

and consequently $(m-l+n-p)(m+l+1)=0$. This proves

$$l=m+n-p \tag{4.5}$$

and proves the

LEMMA 4.1. *If an eigenform of the second kind U in a Sasakian submersion has degree p, order m, coorder l, they satisfy $l+p=n+m$.*

Hence p satisfies $p<n+m+1$. In view of Lemma 3.6 we get

THEOREM 4.2. *Let U be a p-form of order m in the Kaehler manifold (M, g, J) of a Sasakian submersion π: $(\tilde{M}, \tilde{g}, \tilde{\xi}) \rightarrow (M, g, J)$. If U is an eigenform of the second kind, $\tilde{\lambda}-\lambda$ is given by* (3.9), *p satisfies $p<n+m+1$ and* (3.10), (3.11), (3.12) *are valid. $\tilde{\lambda}=\lambda$ if and only if $m=0$. Moreover U_0 in the equation* (3.11) *is uniquely determined.*

Proof. We need only to prove that U_0 is unique. As this is obvious if $m=0$, we assume $m>0$. Moreover it is sufficient to show that, if a $(p-2m)$-form X satisfies

$$\Xi^m X=0, \qquad TX=0,$$

then $X=0$. Now from ${}^{B}(3.5)_k$ we get

$$(p-2m+2k)(p-2m+2k-1)T\Xi^k X$$
$$=4k(n+1-k-p+2m)\Xi^{k-1}X.$$

As p satisfies $n+m+1>p$, we get $n+1-k-p+2m>0$ for $k=m, \quad m-1, \cdots, 1$. Hence beginning with $\Xi^m X=0$ we get successively $\Xi^{m-1}X=0, \cdots, X=0$.

REMARK. If $p=0$ then $m=0$ and we get $\tilde{\lambda}=\lambda$. This case is well-known in any Riemannian submersion with totally geodesic fibers [1].

Hence the $(p-2m)$-form U_0 of (3.11) is $\alpha T^m U$ where α is some positive number. Now we use the

LEMMA 4.3. *If V is an eigenform of a Kaehler manifold, then ΞV and TV are also eigenforms with the same eigenvalue.*

This is the result of the well-know property of Δ in a Kaehler manifold (see Remark in §5)

$$\Delta\Xi=\Xi\Delta, \qquad \Delta T=T\Delta. \tag{4.6}$$

Then we can obtain the

THEOREM 4.4. *Assume that a p-form U of order m is an eigenform of the second kind in a Sasakian submersion. Then the form U_0 satisfying (3.11) is also an eigenform of the second kind. Any form $\Xi^k U$ is also an eigenform of the second kind. Moreover, if U is an eigenform of the first kind, U_0 is also an eigenform of the first kind.*

Proof. From $\Delta U=\lambda U$ and (4.6) we get $\Delta T^m U=T^m\Delta U=\lambda T^m U$, hence $\Delta U_0=\lambda U_0$. As $TU_0=0$ we get $(\tilde{\Delta} U_0{}^L)_{H\cdots H}=\lambda U_0{}^L$ from (2.11) and consequently U_0 is an eigenform of the second kind. ΞU is an eigenform of Δ by Lemma 4.3. As we have $\Xi U=\Xi^{m+1}U_0$, $TU_0=0$, we get in view of ${}^{B}(3.5)$

$$(p+2)(p+1)T\Xi U=(p+2)(p+1)T\Xi^{m+1}U_0$$
$$=4(m+1)(n+m-p)\Xi^m U_0$$
$$=4(m+1)(n+m-p)U.$$

Hence we have

$$(p+2)(p+1)\Xi T(\Xi U)=4(m+1)(n+m-p)\Xi U$$

which proves that ΞU is an eigenform of the second kind. From Lemma 2.2 and (2.12) we get $F^{ts}\nabla_t U_{si_2\cdots i_p}=0$ as a necessary condition for U to be an eigenform of the first kind. From this we get $F^{ts}\nabla_t(U_{srqi_4\cdots i_p}F^{rq})=0$ and so on. Hence U_0 is an eigenform of the first kind if U is an eigenform of the first kind.

We also obtain

THEOREM 4.5. *If a p-form U is an eigenform of the second kind in a Sasakian submersion, then so is also* $*U$.

Proof. As $*\Delta=\Delta*$, $*U$ is also an eigenform. By ${}^{B}(3.6)$ U is at the same time an eigenelement of the linear operator ΞT if $p(p-1)\neq 0$. By ${}^{B}(3.3)_1$ U is an eigenelement of the linear operator $T\Xi$. If $p(p-1)=0$, we directly find that U is an eigenelement of $T\Xi$. This implies that $*U$ is an eigenelement of ΞT, hence an eigenform of the second kind by Proposition 1.2.

COROLLARY 4.6. *Let a p-form U be an eigenform of the second kind and let $\tilde{\lambda}$ and $\tilde{\lambda}_*$ be the eigenvalues of the relative eigenforms U^L and $(*U)^L$ respectively. Then*

$$\tilde{\lambda}_*-\tilde{\lambda}=4(n-p). \tag{4.7}$$

Proof. Let the order of U be m and the coorder of U be l. Then $*U$ is of order l and we get from (3.9)

$$\tilde{\lambda}-\lambda=4m(m+1+n-p),$$

$$\tilde{\lambda}_*-\lambda=4l(l+1+n-(2n-p))=4l(l-n+p+1).$$

From (4.5) we get (4.7).

Now let us consider the problems:

When an eigenform U of Δ is given, how can we construct an eigenform of the second kind?

Can we express an arbitrary eigenform U of Δ as a linear combination of eigenforms of the second kind?

Let U be an eigenform of degree p and order m. According to Lemma 4.3 the following p-forms span a linear subspace of an eigenspace of Δ,

$$U,\ \Xi TU, \cdots, \Xi^m T^{km} U.$$

In view of ${}^{B}(3.3)_k$ we get

$$T\Xi^k T^k U=a_k\Xi^{k-1}T^k U+b_k\Xi^k T^{k+1}U$$

where

$$a_k=4k(n+1-p+k)/p(p-1),$$

$$b_k=(p-2k)(p-2k-1)/p(p-1).$$

Then we get

$$\Xi T(\Xi^k T^k U) = a_k \Xi^k T^k U + b_k \Xi^{k+1} T^{k+1} U. \tag{4.8}$$

Let us define the matrices M and x by

$$M = \left\| \begin{matrix} a_0 & 0 & 0 \cdots\cdots & 0 \\ b_0 & a_1 & 0 \cdots\cdots & 0 \\ 0 & b_1 & a_2 \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 \cdots & 0 & b_{m-1} & a_m \end{matrix} \right\|, \qquad x = \left\| \begin{matrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_m \end{matrix} \right\|.$$

Then we can put

$$\Xi T(x_0 U + x_1 \Xi T U + \cdots + x_m \Xi^m T^m U)$$
$$= y_0 U + y_1 \Xi T U + \cdots + y_m \Xi^m T^m U$$

where $Mx = y$. Assume $p \leqq n$. Then we have $0 = a_0 < a_1 < \cdots < a_m$ so that we get eigenvalues and eigenvectors of the following form

$$a_0(=0): \left\| \begin{matrix} 1 \\ * \\ * \\ \vdots \\ * \\ * \end{matrix} \right\|, \quad a_1: \left\| \begin{matrix} 0 \\ 1 \\ * \\ \vdots \\ * \\ * \end{matrix} \right\|, \cdots, \quad a_m: \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \right\|. \tag{4.9}$$

The eigenform $V(x)$ of Δ where $V(x) = x_0 U + x_1 \Xi T U + \cdots + x_m \Xi^m T^m U$ and x is one of eigenvectors is an eigenelement of the linear operator ΞT. Hence $V(x)$ is an eigenform of the second kind.

From (4.9) we can see that, conversely the given eigenform U is a linear combination of such eigenforms $V(x)$ of the second kind. This proves the following Theorem 4.7 in the case of $p \leqq n$. If $p > n$ we take $*U$ which is also an eigenform of Δ. To complete the proof we only need Theorem 4.5.

THEOREM 4.7. *In a Sasakian submersion any eigenspace of Δ is spanned by some eigenforms of the second kind.*

§ 5. Eigenforms of the first kind.

Now let us turn to eigenforms of the first kind.

THEOREM 5.1. *Let a p-form V be an eigenform of the first kind. Then that ΞV is an eigenform of the first kind is equivalent to that V is a closed form.*

Proof is easily obtained from the identity

$$(p+2) F^{ts} \nabla_t (F_{[s i_0} V_{i_1 \cdots i_p]}) = (p+2) F_{[s i_0} F^{ts} \nabla_{|t|} V_{i_1 \cdots i_p]}$$
$$= -2 \nabla_{[i_0} V_{i_1 \cdots i_p]} + p F_{[i_0 i_1} F^{ts} \nabla_{|t} V_{s| i_2 \cdots i_p]}.$$

THEOREM 5.2. *If a closed form V is an eigenform of the first kind, then $\Xi^k V$, $T^k V$ are closed forms and are eigenforms of the first kind.*

Proof. As V is a closed form, we can see easily that ΞV is a closed form. In view of Theorem 5.1 $\Xi^k V$ is an eigenform of the first kind. As we have already Theorem 4.4 we need only to show that TV is a closed form. This is obtained from the identity

$$(p+1)F^{ts}\nabla_{[t}V_{si_2\cdots i_p]}$$
$$=2F^{ts}\nabla_t V_{si_2\cdots i_p}+(p-1)\nabla_{[i_2}V_{i_3\cdots i_p]ts}F^{ts}.$$

From the same identity we get

LEMMA 5.3. *An eigenform V of order zero is an eigenform of the first kind if V is a closed form.*

Now we get the

THEOREM 5.4. *Any eigenform of the second kind of degree $p\leqq n$ is an eigenform of the first kind if it is a closed form.*

Proof. Let a closed p-form U of order m be an eigenform of the second kind and assume $p\leqq n$. We get $\Xi TU=\alpha U$ where α is a number. As $dU=0$ we get $d\Xi TU=0$, hence $\Xi dTU=0$. We get in view of Lemma 3.3 $dTU=0$, hence TU is a closed eigenform of the second kind of degree $p-2<n$. We can repeat this process until we find that $T^m U$ is a closed eigenform of the second kind of order zero. Hence $T^m U$ is an eigenform of the first kind by Lemma 5.3. This proves the theorem in view of Theorem 5.1.

THEOREM 5.5. *If a p-form $(p\leqq n)$ U of order m is an eigenform of Δ and $U, TU, \cdots, T^m U$ are closed forms, then U is a linear combination of eigenforms of the first kind.*

Proof. If x is an eigenvector of M, $x_0 U+x_1\Xi TU+\cdots+x_m\Xi^m T^m U$ is an eigenform of the second kind and is a closed form, hence an eigenform of the first kind by Theorem 5.4. Taking all eigenvectors of M we can complete the proof.

REMARK As the main object of the present paper is the Sasakian submersion and the underlying Kaehler manifold is naturally treated in the formalism of real tensor analysis, operators such as $\tilde{T}, \tilde{\Xi}$ and then T, Ξ are introduced. T and Ξ are almost the same as the well-known operators Λ and L.

EXAMPLE 1. $\tilde{\Xi}^k\circ\tilde{1}=(\Xi^k\circ 1)^L$ is a closed $2k$-form of order k. As $\tilde{1}$ and 1 are eigenfunctions, 1 is an eigenform of degree zero of the second kind in any

Sasakian submersion. Hence $\Xi^k \circ 1$ is an eigenform of the second kind of degree $2k$ and of order k. The eigenvalue $\tilde{\lambda}$ of $\tilde{\Xi}^k \circ \tilde{1}$ then satisfis $\tilde{\lambda}=4k(n-k+1)$. $\Xi^k \circ l$ is an eigenform of the first kind.

EXAMPLE 2. If $\tilde{f}=f^L$ is and eigenfunction in $(S^{2n+1}, \tilde{g}_0)$ with eigenvalue $\lambda=q(2n+q)$, f is an eigenfunction in (CP^n, g_0) with the same eigenvalue [1]. The derivative ∇f is a closed 1-form and is an eigenform in (CP^n, g_0) with the same eigenvalue. Its order is zero and it is easy to see that ∇f is an eigenform of the first kind. It is easy to see that the $(2k+1)$-form $\Xi^k \nabla f$ is an eigenform of the first kind too, with the same eigenvalue λ as f. The eigenvalue $\tilde{\lambda}$ of the lift $\tilde{\Xi}^k \tilde{\nabla} \tilde{f}$ is given by

$$\tilde{\lambda}-\lambda=4k(n+k+1-2k-1)=4k(n-k),$$

hence $\tilde{\lambda}=q(2n+q)+4nk-4k^2$.

EXAMPLE 3. Let U be a 1-form and assume U and $*U$ are eigenforms of the first kind. Then we can prove that U is a harmonic form, hence $U=0$ if $b_1(M)=0$.

References

[1] M. Berger, P. Gauduchon, and E. Mazet, *Le spectre d'une variété riemannienne*, Springer-Verlag 1971.

[2] D. E. Blair, *Contact manifolds in Riemannian geomentry*, Springer-Verlag 1976.

[3] R. H. Escobales, *Riemannian submersions with totally geodesic fibers*, J. Differential Geometry, **10** (1975), 253-276.

[4] A. Ikeda and Y. Taniguchi, *Spectra and eigenforms of the Laplacian on S^n and $P^n(\mathbf{C})$*, Osaka J. Math., **15** (1978), No. 3, to appear.

[5] S. Ishihara and M. Konishi, *Differential geometry of fibred spaces*, Publications of the study group of geometry, Tokyo, **8**, (1973).

[6] Y. Mutō, *Riemannian submersions and critical Riemannian metrics*, J. Math. Soc. Japan, **29** (1977), 493-511.

[7] Y. Mutō, *Riemannian submersion and the Laplace-Beltrami operator*, Kōdai Math. Sem. Rep., to appear.

[8] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J., **13** (1966), 459-469.

[9] S. Sasaki and Y. Hatakeyama, *On differentiable manifolds with contact metric structures*, J. Math. Soc. Japan, **14** (1962), 249-271.

[10] K. Yano and S. Ishihara, *Differential geometry of fibred spaces*, Kōdai Math. Sem. Rep., **19** (1967), 257-288.

[11] K. Yano and S. Ishihara, *Fibred spaces with invariant Riemannian metric*, Kōdai Math. Sam. Rep., **19** (1967), 317-360.

Yokohama, Japan