

ON THE COSYMPLECTIC BOCHNER CURVATURE TENSOR

BY SANG-SEUP EUM

§ 0. Introduction.

In a previous paper [3] K. Yano, U-H. Ki and the present author studied on transversal hypersurfaces of an almost contact manifold, that is, hypersurfaces which never contain the vector field ξ defining the almost contact structure.

In the present paper, we shall find a relation between the contact Bochner curvature tensor of a Sasakian manifold and the Bochner curvature tensor of the transversal Kaehlerian hypersurface.

The purpose of the present paper is to define the cosymplectic Bochner curvature tensor in relation to the contact Bochner curvature tensor and to seek out the form of the components of the cosymplectic Bochner curvature tensor.

§ 1. Transversal hypersurface of an almost contact manifold.

Let M be $(2n+1)$ -dimensional differentiable manifold covered by a system of coordinate neighborhoods $\{U; y^h\}$, where, here and in the sequel, the indices h, i, j, k, \dots run over the range $\{1, 2, \dots, 2n+1\}$ and let M admit an almost contact metric structure, that is, a set $(\varphi_j^h, \xi^h, \eta_j, g_{ji})$ of a tensor field φ_j^h of type $(1, 1)$, a vector field ξ^h , a 1-form η_j and a positive definite Riemannian metric g_{ji} satisfying

$$(1.1) \quad \varphi_j^h \varphi_i^j = -\delta_i^h + \eta_i \xi^h, \quad \eta_i \varphi_h^i = 0, \quad \varphi_i^h \xi^i = 0, \quad \eta_i \xi^i = 1,$$

$$(1.2) \quad g_{ji} \varphi_h^j \varphi_k^i = g_{hk} - \eta_h \eta_k, \quad \eta_j = g_{ji} \xi^i, \quad g_{ji} \xi^j \xi^i = 1.$$

Consider a $2n$ -dimensional differentiable manifold \bar{M} covered by a system of coordinate neighborhoods $\{V; x^a\}$, where, here and in the sequel, the indices a, b, c, d, \dots run over the range $\{1, 2, \dots, 2n\}$, and assume that \bar{M} is differentially immersed in M as a hypersurface by the immersion $i: \bar{M} \rightarrow M$, which expressed locally by $y^h = y^h(x^a)$.

We assume that for each $p \in \bar{M}$, the vector field ξ^h at $i(p)$ never belongs to the tangent hyperplane of the hypersurface $i(\bar{M})$. We call such a hyper-

surface $i(\bar{M})$ a *transversal hypersurface* of an almost contact manifold. In this case, we can take ξ^h as an affine normal to the hypersurface $i(\bar{M})$.

Now the vector $B_a^h = \partial_a y^h$ ($\partial_a = \partial/\partial x^a$) and ξ^h being linearly independent, the transforms $\varphi_h^i B_a^h$ of B_a^h by φ_h^i can be expressed as

$$(1.3) \quad \varphi_h^i B_a^h = B_b^i F_a^b + \alpha_a \xi^i,$$

where F_a^b is a tensor field of type (1, 1) and α_a a 1-form of \bar{M} .

Applying φ_k^i again to (1.3) and taking account of (1.1), we find

$$(1.4) \quad F_b^c F_a^b = -\delta_a^c, \quad \alpha_c F_a^c = \eta_a,$$

where

$$(1.5) \quad \eta_a = B_a^h \eta_h.$$

Thus \bar{M} admits an almost complex structure F and 1-form α .

Transvecting the first equation of (1.2) with $B_b^h B_c^k$ and using (1.3), we obtain [3]

$$(1.6) \quad F_c^a F_b^e (g_{ae} - \eta_a \eta_e) = g_{cb} - \eta_c \eta_b,$$

where $g_{cb} = B_c^k B_b^j g_{kj}$.

This shows that

$$(1.7) \quad \gamma_{cb} = g_{cb} - \eta_c \eta_b$$

is an almost Hermitian metric with respect to the almost complex structure F .

In the previous paper [3], we proved that γ_{cb} is positive definite.

We now assume that \bar{M} is orientable and choose a unit vector field C^h of M normal to $i(\bar{M})$ in such a way that $2n+1$ vectors B_a^h and C^h give the positive orientation of M . We put

$$(1.8) \quad \xi^h = B_a^h v^a + \lambda C^h,$$

then we have

$$(1.9) \quad g_{cb} \gamma^c \gamma^b = 1 - \lambda^2,$$

where $\eta^c = g^{cb} \eta_b$, from which, we see that the contravariant components γ^{cb} of the metric γ are given by

$$(1.10) \quad \gamma^{cb} = g^{cb} + \frac{1}{\lambda^2} \eta^c \eta^b.$$

Denoting the inverse matrix of the matrix $\begin{pmatrix} B_a^h \\ C^h \end{pmatrix}$ by (B^a_k, C_k) , the following relations are well known.

$$(1.11) \quad B_a^k B^b_k = \delta_a^b, \quad B_a^k C_k = 0, \quad B_a^k C^k = 0, \quad C_k C^k = 1,$$

$$(1.12) \quad B_a^k B^a_j + C^k C_j = \delta_j^k,$$

$$(1.13) \quad g_{cb} B^c_k = g_{kj} B_b^j, \quad g^{bc} B_c^k = g^{kj} B_b^j.$$

We denote by K_{cba}^d the curvature tensor of the Levi-Civita connection formed with g_{cb} and by R_{cba}^d the curvature tensor of the connection formed with γ_{cb} . In this case, the equation of Gauss of $i(\bar{M})$ in M is given by

$$(1.14) \quad K_{cbad} = K_{kjih} B_c^k B_b^j B_a^i B_d^h + (h_{cd} h_{ba} - h_{bd} h_{ca}),$$

where $K_{cbad} = K_{cba}^e g_{ed}$, $K_{kjih} = K_{kji}^t g_{th}$ and h_{cb} is the second fundamental tensor of $i(\bar{M})$, K_{kji}^h being the curvature tensor of the connection formed with g_{ji} of M .

§2. Bochner curvature tensor of a transversal hypersurface of a Sasakian manifold.

In this section we consider the Bochner curvature tensor of a transversal hypersurface $i(\bar{M})$ of a Sasakian manifold M .

In a Sasakian manifold $M(\varphi_j^h, \xi^h, \eta_j, g_{ji})$, we have

$$(2.1) \quad \nabla_k \varphi_j^h = -g_{kj} \xi^h + \delta_k^h \eta_j,$$

$$(2.2) \quad \nabla_k \xi^h = \varphi_k^h,$$

where ∇_k is the operator of covariant differentiation with respect to g_{ji} of M .

In the previous paper [3], we verified the fact that the almost Hermitian structure (F, γ) introduced in a transversal hypersurface $i(\bar{M})$ of a Sasakian manifold M is Kaehlerian. For a transversal hypersurface $i(\bar{M})$ of a Sasakian manifold M , we also verified in the previous paper the fact that

$$\begin{aligned} & R_{cbad} - (\gamma_{cd} \gamma_{ba} - \gamma_{bd} \gamma_{ca} + F_{cd} F_{ba} - F_{bd} F_{ca} - 2F_{cb} F_{ad}) \\ &= (K_{cbad} - h_{cd} h_{ba} + h_{bd} h_{ca}) - (g_{cd} g_{ba} - g_{bd} g_{ca}). \end{aligned}$$

Applying the equation of Gauss of $i(\bar{M})$ in M , we obtain, from above-equation,

$$(2.3) \quad \begin{aligned} & R_{cbad} - (\gamma_{cd} \gamma_{ba} - \gamma_{bd} \gamma_{ca} + F_{cd} F_{ba} - F_{bd} F_{ca} - 2F_{cb} F_{ad}) \\ &= (K_{kjih} - g_{kh} g_{ji} + g_{jh} g_{ki}) B_c^k B_b^j B_a^i B_d^h, \end{aligned}$$

where $R_{cbad} = R_{cba}^e \gamma_{ed}$.

We now consider the so-called contact Bochner curvature tensor [5] in a Sasakian manifold defined by

$$\begin{aligned}
(2.4) \quad B_{kji}{}^h &= K_{kji}{}^h + (\delta_k^h - \eta_k \xi^h) L_{ji} - (\delta_j^h - \eta_j \xi^h) L_{ki} \\
&\quad + L_k^h (g_{ji} - \eta_j \eta_i) - L_j^h (g_{ki} - \eta_k \eta_i) \\
&\quad + \varphi_k^h M_{ji} - \varphi_j^h M_{ki} + M_k^h \varphi_{ji} - M_j^h \varphi_{ki} \\
&\quad - 2(M_{kj} \varphi_i^h + \varphi_{kj} M_i^h) + (\varphi_k^h \varphi_{ji} - \varphi_j^h \varphi_{ki} - 2\varphi_{kj} \varphi_i^h),
\end{aligned}$$

where

$$(2.5) \quad L_{ji} = -\frac{1}{2(n+2)} [K_{ji} + (L+3)g_{ji} - (L-1)\eta_j \eta_i],$$

$$(2.6) \quad K_{ji} = K_{tji}{}^t, \quad L_k^h = L_{kt} g^{th},$$

$$(2.7) \quad M_{ji} = -L_{jt} \varphi_i^t, \quad M_k^h = M_{kt} g^{th},$$

and

$$(2.8) \quad L = g^{ji} L_{ji} = -\frac{K+2(3n+2)}{4(n+1)}, \quad K = K_{ji} g^{ji}.$$

Since the structure (F, γ) introduced in a transversal hypersurface $i(\bar{M})$ of a Sasakian manifold is Kaehlerian [3], the Bochner curvature tensor of $i(\bar{M})$ is defined by [4]

$$\begin{aligned}
(2.9) \quad B_{cba}{}^d &= R_{cba}{}^d + \delta_c^d L_{ba} - \delta_b^d L_{ca} + \gamma_{ba} L_c^d - \gamma_{ca} L_b^d \\
&\quad + F_c^d M_{ba} - F_b^d M_{ca} + M_c^d F_{ba} - M_b^d F_{ca} \\
&\quad - 2(M_{cb} F_a^d + F_{cb} M_a^d),
\end{aligned}$$

where

$$(2.10) \quad L_{bc} = -\frac{1}{2(n+2)} (R_{bc} - \frac{R}{4(n+1)} \gamma_{bc}), \quad L_b^a = L_{bc} \gamma^{ca},$$

$$(2.11) \quad R_{bc} = R_{abc}{}^a, \quad R = R_{bc} \gamma^{bc},$$

and

$$(2.12) \quad M_{bc} = -L_{ba} F_c^a, \quad M_b^a = M_{bc} \gamma^{ca}.$$

We now compute $B_c^k B_b^j B_a^i B_d^h B_{kjih}$, where $B_{kjih} = B_{kji}{}^t g_{th}$.

The equation (2.4) can be written in the covariant form:

$$\begin{aligned}
(2.13) \quad B_{kjih} &= K_{kjih} + (g_{kh} - \eta_k \eta_h) L_{ji} - (g_{jh} - \eta_j \eta_h) L_{ki} \\
&\quad + L_{kh} (g_{ji} - \eta_j \eta_i) - L_{jh} (g_{ki} - \eta_k \eta_i) \\
&\quad + \varphi_{kh} M_{ji} - \varphi_{jh} M_{ki} + M_{kh} \varphi_{ji} - M_{jh} \varphi_{ki} \\
&\quad - 2(M_{kj} \varphi_{ih} + \varphi_{kj} M_{ih}) + (\varphi_{kh} \varphi_{ji} - \varphi_{jh} \varphi_{ki} - 2\varphi_{kj} \varphi_{ih}).
\end{aligned}$$

Transvecting (2.3) with γ^{cd} and taking account of (1.7), (1.10) – (1.13) and the fact that $C_k \xi^k = \lambda$, we obtain

$$(2.14) \quad \begin{aligned} & R_{ba} - 2(n+1)\gamma_{ba} + (2n-1)g_{ba} + \frac{1}{\lambda^2}(\eta_c \eta^c)g_{ba} - \frac{1}{\lambda^2}\eta_b \eta_a \\ & = K_{ji}B_b^j B_a^i + \frac{1}{\lambda^2}K_{kjih}\xi^k B_b^j B_a^i \xi^h - \frac{1}{\lambda}K_{kjih}B_b^j B_a^i (C^k \xi^h + \xi^k C^h). \end{aligned}$$

On the other hand, from (2.2) and the Ricci identity, we obtain

$$(2.15) \quad -K_{kjih}\xi^h = g_{ki}\eta_j - g_{ji}\eta_k,$$

from which

$$(2.16) \quad K_{kjih}\xi^h \xi^k = g_{ji} - \eta_j \eta_i.$$

Substituting (2.15) and (2.16) into (2.14) and taking account of (1.11) and the fact that

$$(2.17) \quad \eta_c \eta^c = 1 - \lambda^2$$

which is obtained from (1.9) and $\eta^a = \xi^k B^a_k$, we have

$$(2.18) \quad R_{ba} - 2(n+1)\gamma_{ba} + 2ng_{ba} = K_{ji}B_b^j B_a^i,$$

from which

$$(2.19) \quad R_{ba} + 2n\eta_b \eta_a - 2\gamma_{ba} = K_{ji}B_b^j B_a^i$$

by the help of (1.7).

Transvecting (2.19) with γ^{ba} and taking account of (1.10) – (1.13) and (2.17), we obtain

$$(2.20) \quad R - 4n + \frac{2n}{\lambda^2}(1 - \lambda^2) = K + \frac{1}{\lambda^2}K_{ji}\xi^j \xi^i - \frac{2}{\lambda}K_{ji}C^j \xi^i,$$

where $R = R_{ba}\gamma^{ba}$.

Transvecting (2.15) with g^{ji} , we obtain

$$(2.21) \quad K_{kh}\xi^h = 2n\eta_k.$$

Substituting (2.21) into (2.20) and taking account of $\eta_j C^j = \lambda$, we obtain

$$(2.22) \quad R - 2n = K.$$

Transvecting (2.5) with $B_b^j B_a^i$ and substituting (2.19) and (2.22), we obtain

$$(2.23) \quad B_b^j B_a^i L_{ji} = L_{ba} - \eta_b \eta_a,$$

where

$$(2.24) \quad L_{ba} = -\frac{1}{2(n+2)} \left[R_{ba} - \frac{R}{4(n+1)} \gamma_{ba} \right].$$

On the other hand, transvecting (2.5) with ξ^i , we get

$$(2.25) \quad -L_{jt}\xi^t = \eta_j.$$

Transvecting φ_{ji} with $B_b^j B_a^i$ and taking account of (1.3), (1.4) and (1.7), we have

$$(2.26) \quad B_b^j B_a^i \varphi_{ji} = F_{ba}.$$

Transvecting (2.7) with $B_b^j B_a^i$ and taking account of (1.4), (2.23) and (2.26), we obtain

$$(2.27) \quad B_b^j B_a^i M_{ji} = M_{ba},$$

where

$$(2.28) \quad M_{ba} = -L_{bt} F_a^t.$$

Transvecting (2.13) with $B_c^k B_b^j B_a^i B_d^h$ and taking account of (2.3), (2.23), (2.26) and (2.27), we obtain

$$(2.29) \quad B_c^k B_b^j B_a^i B_d^h B_{kjih} = B_{cbad},$$

where $B_{cbad} = B_{cba^e} \gamma_{ed}$.

Thus we have the following

THEOREM 2.1. *The contact Bochner curvature tensor of a Sasakian manifold M and the Bochner curvature tensor of a transversal Kaehlerian hypersurface of M are related by (2.29).*

THEOREM 2.2. *If the contact Bochner curvature tensor of a Sasakian manifold M vanishes, then the Bochner curvature tensor of a transversal Kaehlerian hypersurface of M also vanishes.*

§3. Cosymplectic Bochner curvature tensor of a cosymplectic manifold.

An almost contact metric manifold $M(\varphi_j^h, \xi^h, \eta_j, g_{kj})$ is said to be cosymplectic if the 2-form $\Phi_{ji} = \varphi_j^h g_{hi}$ and the 1-form η_j are both closed. It is known [1] that the cosymplectic structure is characterized by

$$(3.1) \quad \nabla_k \varphi_j^h = 0, \quad \nabla_k \xi^h = 0,$$

where ∇_k is the operator of covariant differentiation with respect to g_{ji} of M .

We verified in the previous paper [3] the fact that the almost Hermitian

structure (F, γ) introduced in a transversal hypersurface $i(\bar{M})$ of a cosymplectic manifold M is Kaehlerian.

For a transversal hypersurface $i(\bar{M})$ of a cosymplectic manifold M , we also verified in the previous paper [3] that

$$R_{cba}{}^d = (K_{cbae} - h_{ce}h_{ba} + h_{be}h_{ca})\gamma^{ed}.$$

Thus we have

$$(3.2) \quad R_{cbad} = K_{cbad} - (h_{cd}h_{ba} - h_{bd}h_{ca}),$$

where $R_{cbad} = R_{cba}{}^e\gamma_{ed}$ and $K_{cbad} = K_{cba}{}^e\gamma_{ed}$.

Substituting (3.2) into (1.14), we obtain

$$(3.3) \quad R_{cbad} = K_{kjih}B_c{}^k B_b{}^j B_a{}^i B_d{}^h.$$

In §2, we proved that the contact Bochner curvature tensor B_{kjih} of a Sasakian manifold M and the Bochner curvature tensor of a transversal hypersurface of M are related by (2.29), that is,

$$B_c{}^k B_b{}^j B_a{}^i B_d{}^h B_{kjih} = B_{cbad}.$$

In relation to this fact, we are now going to certify in a cosymplectic manifold M the existence of a tensor \bar{B}_{kjih} satisfying the relation

$$(*) \quad B_c{}^k B_b{}^j B_a{}^i B_d{}^h \bar{B}_{kjih} = B_{cbad},$$

where B_{cbad} is the Bochner curvature tensor of a transversal Kaehlerian hypersurface of M .

The purpose of this section is to seek out the form of the components of a tensor \bar{B}_{kjih} .

The Bochner curvature tensor $B_{cba}{}^d$ of a transversal Kaehlerian hypersurface $i(\bar{M})$ of a cosymplectic manifold M is defined by (2.9).

Transvecting (3.3) with γ^{cd} and taking account of (1.10)-(1.13), we obtain

$$(3.4) \quad R_{ba} = (B^c{}_l B^d{}_m g^{lm} + \frac{1}{\lambda^2} B^c{}_l B^d{}_m \xi^l \xi^m) B_c{}^k B_b{}^j B_a{}^i B_d{}^h K_{kjih},$$

where we have used the relation

$$(3.5) \quad \gamma^a = \xi^k B^a{}_k.$$

Taking account (1.12) and the fact that $C_k \xi^k = \lambda$, we obtain from (3.4),

$$(3.6) \quad R_{ba} = K_{ji} B_b{}^j B_a{}^i + \frac{1}{\lambda^2} \left[\xi^k \xi^h - \frac{1}{\lambda} (C^k \xi^h + C^h \xi^k) \right] B_b{}^j B_a{}^i K_{kjih}.$$

On the other hand, using (3.1) and the Ricci identity:

$$\nabla_k \nabla_j \eta_i - \nabla_j \nabla_k \eta_i = -K_{kji}{}^h \eta_h,$$

we obtain

$$(3.7) \quad K_{kji}{}^h \eta_h = 0,$$

from which,

$$(3.8) \quad K_{kj}{}^c{}^k = 0.$$

Substituting (3.7) into (3.6), we obtain

$$(3.9) \quad R_{ba} = K_{ji} B_b{}^j B_a{}^i.$$

Transvecting (3.9) with γ^{ba} and taking account of (1.7), (1.10)-(1.13) and (3.8), we obtain

$$(3.10) \quad R = K = K_{ji} g^{ji}.$$

Taking account of (1.5), (1.7), (2.10), (3.9) and (3.10), we obtain

$$(3.11) \quad B_b{}^j B_a{}^i L_{ji} = L_{ba},$$

where

$$(3.12) \quad L_{ji} = -\frac{1}{2(n+2)} \left[K_{ji} + L(g_{ji} - \eta_j \eta_i) \right],$$

and

$$(3.13) \quad L = L_{ji} g^{ji} = -\frac{K}{4(n+1)}, \quad K = K_{ji} g^{ji}.$$

Transvecting (3.12) with ξ^i and taking account of (3.8), we obtain

$$(3.14) \quad L_{ji} \xi^i = 0.$$

On the other hand, transvecting φ_{ji} with $B_b{}^j B_a{}^i$ and taking account of (1.3), (1.4) and (1.7) we have

$$(3.15) \quad B_b{}^j B_a{}^i \varphi_{ji} = F_{ba},$$

where $F_{ba} = F_b{}^c \gamma_{ca}$.

By the help of (2.12), (3.11) and (3.15), we obtain

$$(3.16) \quad B_b{}^j B_a{}^i M_{ji} = M_{ba},$$

where

$$(3.17) \quad M_{ji} = -L_{ji} \varphi_i{}^j.$$

Taking account of (3.3), (3.11), (3.15) and (3.16), we obtain

$$\begin{aligned}
 (3.18) \quad & B_c{}^k B_b{}^j B_a{}^i B_d{}^h [K_{kjih} + (g_{kh} - \eta_k \eta_h) L_{ji} - (g_{jh} - \eta_j \eta_h) L_{ki} \\
 & + L_{kh} (g_{ji} - \eta_j \eta_i) - L_{jh} (g_{ki} - \eta_k \eta_i) \\
 & + \varphi_{kh} M_{ji} - \varphi_{jh} M_{ki} + M_{kh} \varphi_{ji} - M_{jh} \varphi_{ki} \\
 & - 2(M_{kj} \varphi_{ih} - \varphi_{kj} M_{ih})] \\
 & = B_{cbad}.
 \end{aligned}$$

Thus we have the following

THEOREM 3.1. *There exists a tensor \bar{B}_{kjih} satisfying the relation (*) in a cosymplectic manifold M ($\dim M = 2n + 1$), and the components of \bar{B}_{kjih} are given by the following:*

$$\begin{aligned}
 \bar{B}_{kjih} = & K_{kjih} + (g_{kh} - \eta_k \eta_h) L_{ji} - (g_{jh} - \eta_j \eta_h) L_{ki} \\
 & + L_{kh} (g_{ji} - \eta_j \eta_i) - L_{jh} (g_{ki} - \eta_k \eta_i) \\
 & + \varphi_{kh} M_{ji} - \varphi_{jh} M_{ki} + M_{kh} \varphi_{ji} - M_{jh} \varphi_{ki} \\
 & - 2(M_{kj} \varphi_{ih} + \varphi_{kj} M_{ih}),
 \end{aligned}$$

where

$$\begin{aligned}
 L_{ji} = & -\frac{1}{2(n+2)} [K_{ji} + L(g_{ji} - \eta_j \eta_i)], \quad L_j{}^h = L_{ji} g^{th}, \\
 L = & L_{ji} g^{ji} = -\frac{K}{4(n+1)}, \quad K = g^{ji} K_{ji}, \\
 M_{ji} = & -L_{ji} \varphi_i{}^t \quad \text{and} \quad M_j{}^h = M_{ji} g^{th}.
 \end{aligned}$$

We call such a tensor \bar{B}_{kjih} the *cosymplectic Bochner curvature tensor* of a cosymplectic manifold. Thus we have the following

THEOREM 3.2. *If the cosymplectic Bochner curvature tensor of a cosymplectic manifold M vanishes, then the Bochner curvature tensor of a transversal Kaehlerian hypersurface of M also vanishes.*

References

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Sung Kyun Kwan University