J. Korean Math. Soc.
Vol. 15, No. 1, 1978.

SOME CHARACTERIZATIONS IN A SASAKIAN MANIFOLD WITH VANISHING C-BOCHNER CURVATURE TENSOR AND ITS SUBMANIFOLDS OF CODIMENSION 2

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1. Introduction.

Many authors have studied on Kaehlerian manifolds with parallel or vanishing Bochner curvature tensor. In 1969 Matsumoto [5] proved

THEOREM A. *If a Kaehlerian manifold M with vanishing (more generally, parallel) Bochner curvature tensor has the constant scalar curvature, then the Ricci tensor of M is parallel and hence M is locally symmetric.*

Furthermore Funabashi [2] has obtained the following theorem.

THEOREM B. *Let M be a real n-dimensional Kaehlerian manifold* ($n=2m$, $m\geq 2$) *with vanishing Bochner curvature tensor. Then the following statements are equivalent*

(1) *M has the constant scalar curvature;*
(2) *M has the parallel Ricci tensor;*
(3) *M is locally symmetric;*
(4) *M satisfies the condition* $K(X, Y)K=0$;
(5) *M satisfies the condition* $K(X, Y)K_1=0$;
(6) *At a point in M, the Ricci tensor has m eigenvalues* $\lambda_\nu(\nu=1, 2, \cdots, m)$ *such that*

$$(\lambda_\mu-\lambda_\nu)[(m+1)(\lambda_\mu+\lambda_\nu)-\Lambda]=0, \quad (\Lambda=\sum_{\rho=1}^{m}\lambda_\rho, \ \ \mu\neq\nu),$$

X and Y being any tangent vectors and K and K_1 *denoting the curvature tensor of M and the Ricci tensor respectively, and where the endomorphism* $K(X, Y)$ *operates on K or* K_1 *as a derivation of tensor algebra at each point in M.*

One of the purpose of this paper is to prove the following Theorem 1 and Corollary corresponding to Theorem B, replacing the vanishing of the Bochner curvature tensor by the parallel of C-Bochner curvature tensor in a Sasakian manifold.

Received by the editors Mar. 31, 1978.

THEOREM 1. *Let M be a Sasakian manifold of dimension n ($\geqq 5$) with parallel C-Bochner curvature tensor. Then the following statements are equivalent:*

(1) *M is an Einstein space;*

(2) *M is locally symmetric;*

(3) *M has the parallel Ricci tensor;*

(4) *M satisfies the condition $K(X, Y)K=0$;*

(5) *M satisfies the condition $K(X, Y)K_1=0$.*

COROLLARY 2. *Let M be a Sasakian manifold of dimension n (≥ 5) with vanishing C-Bochner curvature tensor. Then the statements (1)$\sim$(5) in Theorem 1 are equivalent.*

On the other hand Yano and Ki [11] showed that submanifolds of codimension 2 of Sasakian manifolds admit Sasakian structure under a certain condition. In this point of view we shall prove the following Theorem 3 and Corollary 4.

THEOREM 3. *Let M^{2m+3} be a Sasakian manifold with vanishing C-Bochner curvature tensor. Then there exist no submanifolds of codimension 2 satisfying $\lambda^2+\mu^2+\nu^2=1$ globally except totally geodesic, where λ, μ, ν are differentiable functions defined by (4.3)$\sim$(4.5).*

COROLLARY 4. *Let M^{2m+1} be a submanifold of codimension 2 satisfying $\lambda^2+\mu^2+\nu^2=1$ glovally in $(2m+3)$-dimensional Sasakian manifold M^{2m+3} with vanishing C-Bochner curvature tensor. Then M^{2m+1} is a Sasakian manifold with vanishing C-Bochner curvature tensor, where $2m+1\geq 5$.*

In §2, we recall some fundamental properties concerning with the C-Bochner curvature tensor. In §3, we prove Theorem 1 and Corollary 2. In §4, we develope the structure equations of the Sasakian submanifold M^{2m+1} of codimension 2 satisfing $\lambda^2+\mu^2+\nu^2=1$ in a Sasakian manifold M^{2m+3} (see also [11]). In the last §5, we consider the Sasakian submanifold M^{2m+1} with vanishing C-Bochner curvature tensor and devote to prove Theorem 3 and Corollary 4.

2. Fundamental properties of a Sasakian manifold with parallel C-Bochner curvature tensor.

Let M be an n ($=2m+1\geq 5$)-dimensional Sasakian manifold (or normal contact metric manifold). Then there exists a unit Killing vector field p^h satisfing

$$p_j=g_{jh}p^h, \qquad f_{ji}=\nabla_j p_i, \qquad f_{ji}=-f_{ij},$$

$$(2.1)\qquad \nabla_j f_{ih} = p_i g_{jh} - p_h g_{ji},$$
$$f_j{}^i f_i{}^h = -\delta_j^h + p_j p^h, \qquad f_j{}^i = g^{ih} f_{jh}, \qquad f_j{}^i p^j = 0,$$

where g_{ji} denotes the metric tensor and ∇ denotes the covariant derivative with respect to the Riemannian connection and $f_j{}^i$ denotes (1, 1) type structure tensor.

Recently in an n-dimensional Sasakian manifold M, Matsumoto and Chūman [6] introduced C-Bochner curvature tensor $B_{kji}{}^h$ defined by

$$(2.2)\qquad B_{kji}{}^h = K_{kji}{}^h + \frac{1}{n+3}(K_{ki}\delta_j^h - K_{ji}\delta_k^h + g_{ki}K_j{}^h - g_{ji}K_k{}^h + S_{ki}f_j{}^h - S_{ji}f_k{}^h$$
$$+ f_{ki}S_j{}^h - f_{ji}S_k{}^h + 2S_{kj}f_i{}^h + 2f_{kj}S_i{}^h - K_{ki}p_jp^h + K_{ji}p_kp^h - p_kp_iK_j{}^h)$$
$$-\frac{k+n-1}{n+3}(f_{ki}f_j{}^h - f_{ji}f_k{}^h + 2f_{kj}f_i{}^h) - \frac{k-4}{n+3}(g_{ki}\delta_j^h - g_{ji}\delta_k^h)$$
$$+\frac{k}{n+3}(g_{ki}p_jp^h + p_kp_i\delta_j^h - g_{ji}p_kp^h - p_jp_i\delta_k^h),$$

where the aggregate $(f_j{}^i, p^i, g_{ji})$ is the structure given by (2.1), $K_{kji}{}^h$ the curvature tensor, K the scalar curvature, $S_{kj} = f_k{}^s K_{sj}$, $S_k{}^i = g^{ji} S_{kj}$ and $k = (K + n - 1)/(n+1)$. In fact C-Bochner curvature tensor is the horizontal lift of the Bochner curvature tensor in a Kaehlerian manifold by the fibering of Boothby -Wang [1]. By straightforward computations the following identities are obtained with the help of (2.1):

$$(2.3)\qquad \begin{aligned} &B_{kji}{}^h = -B_{jki}{}^h, \qquad B_{kjih} = B_{ihkj}, \qquad B_{kji}{}^h + B_{jik}{}^h + B_{ikj}{}^h = 0, \\ &B_{kji}{}^k = 0, \qquad B_{kji}{}^h p_h = 0, \qquad f_k{}^s B_{sji}{}^h = f_j{}^s B_{ski}{}^h, \qquad f^{kj} B_{kji}{}^h = 0, \end{aligned}$$

where $B_{kjih} = g_{hs} B_{kji}{}^s$.

On the other hand $f_k{}^s K_{sj} = -f_j{}^s K_{sk}$ and the differential form $S = \frac{1}{2} S_{ji} dx^j \wedge dx^i$ is closed. Therefore we can easily verify that the following equations hold good:

$$S_{ji} = -S_{ij}, \qquad \nabla_k S_j{}^k = \frac{1}{2} f_j{}^k \nabla_k K + (K - n + 1) p_j,$$
$$(2.4)\qquad \nabla_k S_{ji} = p_j K_{ik} - (n-1) g_{jk} p_i + f_j{}^t \nabla_k K_{ti},$$
$$f_j{}^t \nabla_t S_{ik} = -p_j S_{ki} + (n-1) f_{ij} p_k + f_j{}^r f_i{}^s \nabla_r K_{sk},$$
$$\nabla_k K_{ji} - \nabla_j K_{ki} = -f_i{}^r \nabla_r S_{kj} - 2S_{kj} p_i$$
$$+ (n-1)(f_{ki} p_j - f_{ji} p_k + 2 f_{kj} p_i),$$

where we have used $K_{ji}p^i=(n-1)p_j$ (cf. [6], [7], [8]).

Differentiating (2.2) covariantly and using (2.4), we have

$$(2.5)\quad (n+3)\nabla_t B_{kji}{}^t=(n+2)(\nabla_k K_{ji}-\nabla_j K_{ki})-f_k{}^r f_j{}^s(\nabla_r K_{si}-\nabla_s K_{ri})$$
$$+2f_i{}^s f_k{}^r\nabla_s K_{rj}+p^r(p_k\nabla_r K_{ji}-p_j\nabla_r K_{ki})-(n+2)p_k S_{ji}+np_j S_{ki}$$
$$+2(n+1)p_i S_{kj}+\frac{1}{n+1}(g_{ki}p_j-g_{ji}p_k)p^r\nabla_r K+\frac{n-1}{2(n+1)}\{(g_{ki}-p_k p_i)\nabla_j K$$
$$-(g_{ji}-p_j p_i)\nabla_k K+(f_{ki}f_j{}^r-f_{ji}f_k{}^r+2f_{kj}f_i{}^r)\nabla_r K\}$$
$$+(n+1)\{(n+2)p_k f_{ji}-np_j f_{ki}-2(n+1)p_i f_{kj}\}.$$

Transvecting (2.5) with $f_l{}^k f_m{}^j$ and adding the resulting equation to (2.5), we obtain

$$\nabla_t B_{lmi}{}^t+f_l{}^k f_m{}^j\nabla_t B_{kji}{}^t=(\nabla_l K_{mi}-\nabla_m K_{li})-f_l{}^k f_m{}^j(\nabla_k K_{ji}-\nabla_j K_{ki})$$
$$+(n-1)(p_l f_{mi}-p_m f_{li})-p_l S_{mi}+p_m S_{li}+\frac{1}{2(n+3)}(g_{li}p_m-g_{mi}p_l)p^t\nabla_t K.$$

On the other hand, using (2.3), we have

$$f_l{}^k f_m{}^j\nabla_t B_{kji}{}^t=-\nabla_t B_{mli}{}^t,$$

from which,

$$\nabla_k K_{ji}-\nabla_j K_{ki}-f_k{}^r f_j{}^s(\nabla_r K_{si}-\nabla_s K_{ri})-p_k S_{ji}+p_j S_{ki}$$
$$+\frac{1}{2(n+3)}(g_{ki}p_j-g_{ji}p_k)p^r\nabla_r K+(n-1)(p_k f_{ji}-p_j f_{ki})=0.$$

Contracting the last equation with p^k and $p^k g^{ji}$, we find respectively

$$(2.6)\qquad p^t\nabla_t K=0,\qquad p^t\nabla_t K_{ji}=0,$$

from which,

$$\frac{n+3}{n-1}\nabla_t B_{kji}{}^t=\nabla_k K_{ji}-\nabla_j K_{kj}-p_k\{S_{ji}-(n-1)f_{ji}\}+p_j\{S_{ki}-(n-1)f_{ki}\}$$
$$+2p_i\{S_{kj}-(n-1)f_{kj}\}+\frac{1}{2(n+1)}\{(g_{ki}-p_k p_i)\delta_j^t-(g_{ji}-p_j p_i)\delta_k^t$$
$$+f_{ki}f_j{}^t+2f_{kj}f_i{}^t\}\nabla_t K.$$

Thus, in a Sasakian manifold with parallel C-Bochner curvature tensor, we get

$$(2.7)\qquad \nabla_k K_{ji}-\nabla_j K_{ki}=p_k\{S_{ji}-(n-1)f_{ji}\}-p_j\{S_{ki}-(n-1)f_{ki}\}$$
$$-2p_i\{S_{kj}-(n-1)f_{kj}\}-\frac{1}{2(n+1)}\{(g_{ki}-p_k p_i)\delta_j^t$$

$$-(g_{ji}-p_jp_i)\delta_k^t+f_{ki}f_j^t-f_{ji}f_k^t+2f_{kj}f_i^t\}\nabla_tK,$$

(2.8) $$\nabla_kS_{ji}=p_jK_{ki}-p_iK_{kj}+\frac{1}{2(n+1)}\{f_{jk}\delta_i^t-f_{ik}\delta_j^t+2f_{ji}\delta_k^t+(g_{ik}-p_ip_k)f_j^t$$
$$-(g_{kj}-p_kp_j)f_i^t\}\nabla_tK$$

(see also [6], [8]).

Now, we are going to compute ∇_kK_{ji} by using (2.6), (2.7) and (2.8). Differentiating covariantly $S_{ji}=f_j^tK_{ti}$ gives

$$\nabla_kS_{ji}=p_jK_{ki}-(n-1)p_ig_{kj}+f_j^t\nabla_kK_{ti},$$

which together with (2.8) implies

(2.9) $$f_j^t\nabla_kK_{ti}=(n-1)\ p_ig_{kj}-p_iK_{kj}+\frac{1}{2(n+1)}\{f_{jk}\delta_i^t-f_{ik}\delta_j^t+2f_{ji}\delta_k^t$$
$$+(g_{ik}-p_ip_k)f_j^t-(g_{jk}-p_jp_k)f_i^t\}\nabla_tK.$$

Transvecting (2.9) with f_l^j and using (2.6) and (2.7) give

(2.10) $$\nabla_kK_{ji}=-p_j\{S_{ki}-(n-1)f_{ki}\}-p_i\{S_{kj}-(n-1)f_{kj}\}$$
$$-\frac{1}{2(n+1)}\{(-g_{kj}+p_kp_j)\delta_i^t-f_{ik}f_j^t+2(-g_{ji}+p_jp_i)\delta_k^t$$
$$-(g_{ik}-p_ip_k)\delta_j^t-f_{jk}f_i^t\}\nabla_tK$$

(see also [8]).

3. Proofs of Theorem 1 and Corollary 2.

We now assume that the C-Bochner curvature tensor is parallel. Differentiating (2.2) covariantly and using $\nabla_lB_{kji}{}^h=0$, (2.1), (2.9) and (2.10), we obtain

(3.1) $$\nabla_lK_{kji}{}^h$$
$$=-\frac{1}{n+3}[-p_k\{S_{li}-(n-1)f_{li}\}+p_i\{S_{lk}-(n-1)f_{lk}\}$$
$$+\frac{1}{2(n+1)}\{-g_{kl}+p_kp_l)\delta_i^t-f_{il}f_k^t+2(-g_{ki}+p_kp_i)\delta_l^t-(g_{il}-p_ip_l)\delta_k^t$$
$$-f_{kl}f_i^t\}\nabla_tK]\delta_j^h-(\nabla_lK_{ji})\delta_k^h+g_{ki}\nabla_lK_j{}^h-g_{ji}\nabla_lK_{kj}+(\nabla_lS_{ki})f_j{}^h$$
$$+S_{ki}(p_j\delta_l^h-p^hg_{lj})-(\nabla_lS_{ji})f_k^h-S_{ji}(p_k\delta_l^h-p^hg_{lk})$$
$$+(p_kg_{li}-p_ig_{lk})S_i^h+f_{ki}\nabla_lS_j{}^h-(p_jg_{li}-p_ig_{lj})S_k^h-f_{ji}\nabla_lS_k^h$$

$$+2S_{kj}(p_i\delta_l^h-p^hg_{li})+2(\nabla_lS_{kj})f_i^h+2(p_kg_{lj}-p_jg_{lk})S_{ij}+2f_{kj}\nabla_lS_i^h$$
$$-(\nabla_lK_{ki})p_jp^h-K_{ki}f_{lj}p^h-K_{ki}p_jf_l^h+(\nabla_lK_{ji})p_kp^h+K_{ji}f_{lk}p^h+K_{ji}p_kf_{lj}$$
$$-f_{lk}p_iK_j^h-p_kf_{li}K_j^h-p_kp_i\nabla_lK_j^h+f_{lj}p_iK_k^h+p_jf_{li}K_k^h+p_jp_i\nabla_lK_k^h]$$
$$+\frac{1}{n+3}(\nabla_lk)(f_{ki}f_j^h-f_{ji}f_k^h+2f_{kj}f_i^h)+\frac{k+n-1}{n+3}[(p_kg_{li}-p_ig_{lk})f_j^h$$
$$+f_{ki}(p_j\delta_l^h-p^hg_{lj})-(p_jg_{li}-p_ig_{lj})f_k^h-f_{ji}(p_k\delta_l^h-p^hg_{lk})$$
$$+2(p_kg_{lj}-p_jg_{lk})f_i^h+2f_{kj}(p_i\delta_l^h-p^hg_{li})]+\frac{1}{n+3}(\nabla_lk)(g_{ki}\delta_j^h-g_{ji}\delta_k^h)$$
$$-\frac{1}{n+3}(\nabla_lk)(g_{ki}p_jp^h+p_kp_i\delta_j^h-g_{ji}p_kp^h-p_jp_i\delta_k^h)-\frac{k}{n+3}(g_{ki}f_{lj}p^h$$
$$+g_{ki}p_jf_l^h+f_{lk}p_i\delta_j^h+p_kf_{li}\delta_j^h-g_{ji}f_{lk}p^h-g_{ji}p_kf_l^h-f_{lj}p_i\delta_k^h-p_jf_{li}\delta_k^h).$$

We suppose that the manifold M is locally symmetric, that is, $\nabla_lK_{kji}{}^h=0$. Then we can find immediately from $\nabla_lK_{kji}{}^h=0$

$$\nabla_lK_{ji}=0 \quad \text{and} \quad \nabla_lK=0,$$

which together with (2.10) yields

$$S_{ki}=(n-1)f_{ki},$$

or equivalently

$$f_k{}^sK_{si}=(n-1)f_{ki}.$$

Transvecting the last equation with $f_j{}^k$ and taking account of (2.1) and $K_{ji}p^i=(n-1)p_j$, we have

$$K_{ji}=(n-1)g_{ji}.$$

Hence the manifold M is an Einstein space.

Conversely, if M is an Einstein space, then the Ricci tensor K_{ji} of M has the form

$$K_{ji}=(n-1)g_{ji}$$

with the help of $K_{ji}p^i=(n-1)p_j$. Consequently

$$S_{ki}=(n-1)f_{ki}, \qquad k=n-1,$$
(3.2) $$\nabla_kK_{ji}=0, \qquad \nabla_kK=0,$$
$$\nabla_kS_{ji}=(n-1)(p_jg_{ki}-p_ig_{kj}).$$

Substituting (3.2) into (3.1), we can easily see that $\nabla_lK_{kji}{}^h=0$ and consequently M is locally symmetric. Thus the equivalence (1)$\rightleftarrows$(2) is established.

The equivalence (1)$\rightleftarrows$(3) is easily proved by using (2.10) because $\nabla_k K_{ji}=0$ implies $\nabla_k K=0$.

Now we differentiate (2.10) covariantly along M. Then we have

$$\begin{aligned}\nabla_l\nabla_k K_{ji}= & -f_{lj}\{S_{ki}-(n-1)f_{ki}\}-f_{li}\{S_{kj}-(n-1)f_{kj}\}-p_j\{\nabla_l S_{ki}-(n-1)\nabla_l f_{ki}\}\\ & -p_i\{\nabla_l S_{kj}-(n-1)\nabla_l f_{kj}\}-\frac{1}{2(n+1)}\{f_{lj}p_k\delta_i^t+p_j f_{lk}\delta_i^t-(\nabla_l f_{ik})f_i^t\\ & -f_{ik}(\nabla_l f_j^t)+2(f_{lj}p_i+p_j f_{li})\delta_k^t+(f_{li}p_k+p_i f_{lk})\delta_j^t-(\nabla_l f_{jk})f_i^t\\ & -f_{jk}\nabla_l f_i^t\}\nabla_t K-\frac{1}{2(n+1)}\{(-g_{jk}+p_j p_k)-f_{ik}f_j^t+2(-g_{ji}+p_j p_i)\delta_k^t\\ & -(g_{ik}-p_i p_k)\delta_j^t-f_{jk}f_i^t\}\nabla_l\nabla_t K,\end{aligned}$$

from which, using (2.1), (2.4) and (2.6), we find

$$\begin{aligned}(\nabla_l\nabla_k K_{ji}-\nabla_k\nabla_l K_{ji})p^j p^k= & -\{\nabla_l S_{ki}-\nabla_k S_{li}-(n-1)(\nabla_l f_{ki}-\nabla_k f_{li})\}p^k,\\ & +p_i\{\nabla_k S_{lj}-(n-1)\nabla_k f_{lj}\}p^j p^k\end{aligned}$$

because $p^j p^i\nabla_j\nabla_i K=0$. Hence, if $K(X,Y)\cdot K_1=0$, that is, $\nabla_l\nabla_k K_{ji}-\nabla_k\nabla_l K_{ji}=0$, we obtain

(3.3) $$(\nabla_l S_{ki})p^k=p^k\nabla_k S_{li}+(n-1)(g_{li}-p_l p_i)+p_i p^j p^k\nabla_k S_{lj}.$$

On the other hand (2.8) gives

(3.4) $$p^k\nabla_l S_{ki}=K_{li}-(n-1)p_l p_i$$

and

(3.5) $$p^k\nabla_k S_{li}=0$$

Substituting (3.4) and (3.5) into (3.3), we get

$$K_{li}=(n-1)g_{li},$$

which means that the submanifold M is an Einstein space.

When M is an Einstein space, the Ricci tensor is parallel and consequently $K(X,Y)\cdot K_1=0$ holds good. Hence we have (1)$\rightleftarrows$(5).

Finally we shall prove (1)$\rightleftarrows$(4). Let M satisfy the condition $K(X,Y)\cdot K=0$, that is,

$$\nabla_l\nabla_m K_{kji}{}^h-\nabla_m\nabla_l K_{kji}{}^h=0.$$

Then $K(X,Y)\cdot K=0$ implies $K(X,Y)\cdot K_1=0$. Hence M is an Einstein space.

Conversely, if M is an Einstein space, then M is locally symmeric, which gives $K(X, Y)\cdot K=0$. Therefore (1)$\rightleftarrows$(4) is completed. Hence we complete the proofs of Theorem 1 and Corollary 2.

4. Sasakian submanifolds in a Sasakian manifold.

Let M^{2m+3} be a $(2m+3)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{U; x^A\}$ and denote by G_{CB}, $F_B{}^A$ and v^A the components of Riemannian metric tensor, those of the Sasakian structure tensor and the Sasakian structure vector field of M^{2m+3} respectively, where here and in the sequel the indices $A, B, C, D, E, \cdots$ run over the range $\{1, 2, \cdots, 2m+3\}$. Then we have by the definition

$$F_C{}^B F_B{}^A=-\delta_C{}^A+v_C v^A, \qquad F_C{}^A v^C=0, \qquad v_A F_C{}^A=0,$$

$$F_C{}^A F_B{}^D G_{AD}=G_{CB}-v_C v_B, \tag{4.1}$$

$$\nabla_C F_D{}^E=v_D\delta_C{}^E-v^E G_{CD}, \qquad \nabla_C v^D=F_C{}^D,$$

where $v_B=v^A G_{AB}$ and denoting by ∇_C the Riemannian connection with respect to the Christoffel symbols $\{{}_B{}^A{}_C\}$ formed with G_{CB}.

Now we consider a $(2m+1)$-dimensional submanifold M^{2m+1} in a Sasakian manifold M^{2m+3} which is covered by a system of coordinate neighborhoods $\{V; y^h\}$ and immersed isometrically in M^{2m+3} by the immersioin i : $M^{2m+1}\rightarrow M^{2m+3}$, where here and in the sequel the indices $h, i, j, k, g, t, \cdots$run over the range $\{1, 2, \cdots, 2m+1\}$. In the sequel we identify $i(M^{2m+1})$ with M^{2m+1} itself and represent the immersion by $x^A=x^A(y^h)$ locally. We put $B_i{}^A=\partial_i x^A$, $\partial_i=\partial/\partial y^i$ and denote by C^A and D^A two mutually orthogonal unit normals to M^{2m+1}.

If we denote by g_{ji} the fundamental metric tensor of M^{2m+1}, then we have $g_{ji}=B_j{}^C B_i{}^B G_{CB}$ because the immersion is isometric. As to the transforms of $B_i{}^A$, C^A and D^A by $F_B{}^A$ we have equations of the form respectively

$$F_B{}^A B_i{}^B=f_i{}^t B_t{}^A+w_i C^A+u_i D^A, \tag{4.2}$$

$$F_B{}^A C^B=-w^t B_t{}^A-\mu D^A, \tag{4.3}$$

$$F_B{}^A D^B=-u^t B_t{}^A+\mu C^A, \tag{4.4}$$

where $f_i{}^t$ is a tensor field of type (1, 1), u_i, v_i, w_i 1-forms and μ a function in M^{2m+1}, u^i and w^i being the vector fields associated with u_i, w_i respectively. On the other hand the vector field v^A is expressed as a linear combination of $B_i{}^A$, C^A and D^A. Therefore we can put

$$v^A=v^t B_t{}^A+\lambda C^A-\upsilon D^A, \tag{4.5}$$

where v^t is a vector field and λ, ν functions in M^{2m+1}.

Applying the operator F to the both sides of (4.2), (4.3), (4.4) and (4.5), and comparing the tangent and normal parts respectively, we can easily find

$$f_i^t f_t^h = -\delta_i^h + u_i u^h + v_i v^h + w_i w^h, \tag{4.6}$$

$$\begin{aligned} u_t f_i^t &= -\nu v_i + \mu w_i, \\ v_t f_i^t &= \nu u_i - \lambda w_i, \\ w_t f_i^t &= -\mu u_i + \lambda v_i \end{aligned} \tag{4.7}$$

$$\begin{aligned} f_t^h u^t &= \nu v^h - \mu w^h, \\ f_t^h v^t &= -\nu u^h + \lambda w^h \\ f_t^h w^t &= \mu u^h - \lambda v^h, \end{aligned} \tag{4.8}$$

$$\begin{aligned} u_t u^t = 1 - \mu^2 - \nu^2, \quad u_t v^t = \lambda\mu, \quad u_t w^t = \lambda v, \\ v_t v^t = 1 - \nu^2 - \lambda^2, \quad v_t w^t = \mu\nu, \quad w_t w^t = 1 - \lambda^2 - \mu^2. \end{aligned} \tag{4.9}$$

Also, from (4.1) and (4.2), we find

$$f_j^t f_i^s g_{ts} = g_{ji} - u_j u_i - v_j v_i - w_j w_i. \tag{4.10}$$

Equations (4.6), (4.7), (4.8), (4.9) and (4.10) show that the aggregate $(f, g, u, v, w, \lambda, \mu, \nu)$ defines the so-called $(f, g, u, v, w, \lambda, \mu, \nu)$-structure on M^{2m+1}(see [4], [11]).

On the other side the equations of Gauss and those of Weingarten are given by

$$\nabla_j B_i^A = \partial_j B_i^A + \{_B{}^A{}_C\} B_j^B B_i^C - \{_j{}^k{}_i\} B_k^A = h_{ji} C^A + k_{ji} D^A, \tag{4.11}$$

$$\nabla_i C^A = \partial_i C^A + \{_B{}^A{}_C\} B_i^B C^C = -h_i^t B_t^A + l_i D^A, \tag{4.12}$$

$$\nabla_i D^A = \partial_i D^A + \{_B{}^A{}_C\} B_i^B D^C = -k_i^t B_t^A - l_i C^A, \tag{4.13}$$

where $h_i^t = h_{is} g^{st}$, $k_i^t = k_{is} g^{st}$ are second fundamental tensors and l_i third fundamental tensor.

Now we assume that $\lambda^2 + \mu^2 + \nu^2 = 1$ globally on M^{2m+1}. If we put a vector p on M^{2m+1}

$$p^h = \lambda u^h + \mu v^h + \nu w^h. \tag{4.1 }$$

Then the set (f, g, p) defines an almost contact metric structure (cf. [11]):

$$f_j^t f_t^h = -\delta_i^h + p_i p^h, \quad p_t f_i^t = 0, \quad p_t p^t = 1, \quad f_j^t f_i^s g_{ts} = g_{ji} - p_j p_i, \tag{4.15}$$

where $p_i = g_{hi}p^h$.

By means of our assumption, we may consider the following three cases:

(1) $\lambda=0$, $\mu=1$, $\upsilon=0$; (2) $\lambda=1$, $\mu=0$, $\upsilon=0$; (3) $\lambda=0$, $\mu=0$, $\upsilon=1$.

In the case (1), taking account of (4.9) and (4.14), we have $p=v$, $u=o$ and $w=o$, which and (4.2)$\sim$(4.5) and (4.8) imply

$$F_B{}^A B_i{}^B = f_i{}^t B_t{}^A, \tag{4.16}$$

$$F_B{}^A C^B = D^A, \tag{4.17}$$

$$F_B{}^A D^B = -C^A, \tag{4.18}$$

$$v^A = p^t B_t{}^A. \tag{4.19}$$

Differentiating (4.15) covariantly along M, we obtain

$$B_j{}^D(\nabla_D F_B{}^A) B_i{}^B + F_B{}^A(\nabla_j B_i{}^B) = (\nabla_j f_i{}^t) B_t{}^A + f_i{}^t(\nabla_j B_t{}^A),$$

from which, substituting (4.11) and using (4.16), (4.17) and (4.18), we have

$$\nabla_j f_i{}^t = -g_{ji}p^t + p_i\delta_j{}^t, \tag{4.20}$$

$$h_{ji} = k_{jt}f_i{}^t, \quad k_{ji} = -h_{jt}f_i{}^t. \tag{4.21}$$

Similarly differentiating (4.18) covariantly and using (4.11) and (4.15), we have

$$\nabla_j p^t = f_j{}^t, \tag{4.22}$$

$$h_{jt}p^t = 0, \quad k_{jt}p^t = 0. \tag{4.23}$$

Using (4.21) and (4.23), we can easily see that

$$h_{jt}h_i{}^t = k_{jt}k_i{}^t, \quad h_j{}^t k_{ti} + k_j{}^t h_{ti} = 0, \tag{4.24}$$

$$h_t{}^t = 0 = k_t{}^t. \tag{4.25}$$

By the way, in the case (2), from (4.5), (4.9) and (4.12) we find $f_{ji} = -h_{ji}$. But it is contradiction. Similarly, the case (3) can not also occur. Thus we have

LEMMA 1. *Let M^{2m+1} be a submanifold of codimension 2 in a Sasakian manifold M^{2m+3}. If the induced $(f, g, u, v, w, \lambda, \mu, \upsilon)$-structure satisfies $\lambda^2+\mu^2+\upsilon^2=1$ globally, then M^{2m+1} is a minimal Sasakian submanifold (see also [11])*

5. Proofs of Theorem3 and Corollary 4.

In this section we assume that the Sasakian manifold M^{2m+3} is of vanishing

C-Bochner curvature tensor. As already shown in (2.2), the curvature tensor $K_{DCB}{}^{A}$ of M^{2m+3} is the form

$$(5.1)\quad -K_{DCB}{}^{A}=\frac{1}{2(m+3)}(K_{DB}\delta_C{}^A-K_{CB}\delta_D{}^A+G_{DB}K_C{}^A-G_{CB}K_D{}^A+S_{DB}F_C{}^A$$
$$-S_{CB}F_D{}^A+F_{DB}S_C{}^A-F_{CB}S_D{}^A+2S_{DC}F_B{}^A+2F_{DC}S_B{}^A-K_{DB}v_Cv^A$$
$$+K_{CB}v_Dv^A+v_Cv_BK_D{}^A-v_Dv_BK_C{}^A)-\frac{k+2(m+1)}{2(m+3)}(F_{DB}F_C{}^A-$$
$$F_{CB}F_D{}^A+2F_{DC}F_B{}^A)-\frac{k-4}{2(m+3)}(G_{DB}\delta_C{}^A-G_{CB}\delta_D{}^A)$$
$$+\frac{k}{2(m+3)}(G_{DB}v_Cv^A+v_Dv_B\delta_C{}^A-G_{CB}v_Dv^A-v_Cv_B\delta_D{}^A),$$

where $S_{DA}=F_D{}^EK_{EA}$, $S_D{}^A=S_{DE}G^{EA}$ and

$$(5.2)\qquad k=\frac{K+2(m+1)}{2(m+2)}.$$

On the other hand the Gauss equation are given by

$$(5.3)\qquad K_{kjih}=B_k{}^D{}_j{}^C{}_i{}^B{}_h{}^AK_{DCBA}+h_{kh}h_{ji}-h_{jh}h_{ki}+k_{kh}k_{ji}-k_{jh}k_{ki},$$

whereK_{kjih} is the curvature tensor of M^{2m+1} and $B_k{}^D{}_j{}^C{}_i{}^B{}_h{}^A=B_k{}^DB_j{}^CB_i{}^BB_h{}^A$.

Transvecting 4.16) and (4.17) with K_{AD} we get respectively

$$(5.4)\qquad C^AS_{AD}=D^AK_{AD},$$

$$(5.5)\qquad D^AS_{AD}=-C^AK_{AD},$$

from which, transvecting with D^D and C^D we obtain respectively

$$(5.6)\qquad C^AD^DS_{AD}=D^AD^DK_{AD},$$

$$(5.7)\qquad D^AC^DS_{AD}=-C^AC^DK_{AD},$$

which imply

$$(5.8)\qquad K(C,C)=K(D,D),$$

where here and in the sequel we denote by $K(C,C)=C^AC^DK_{AD}$, $K(D,D)=D^AD^DK_{AD}$.

Transvecting g^{ji} to (5.3) and taking account of (4.25) and (5.8), we find

$$(5.9)\qquad \tilde{K}_{kh}=B_k{}^DB_h{}^AK_{DA}-2(B_k{}^DC^CC^BB_h{}^AK_{DCBA}+H_{kh}),$$

where $H_{kh}=h_{ki}h_h{}^i$ and $\tilde{K}_{kh}=g^{ji}K_{kjih}$.

We transvect (4.15) with K_{AC}. Then we get

$$B_i{}^A S_{AC} = f_i{}^t B_t{}^A K_{AC}, \tag{5.10}$$

from which, transvecting with C^C and D^C, we have

$$B_i{}^A C^C S_{AC} = f_i{}^t B_t{}^A D^C K_{AC}, \tag{5.11}$$

$$B_i{}^A D^C S_{AC} = f_i{}^t B_t{}^A D^C K_{AC} \tag{5.12}$$

respectively.

We consider the Sasakian submanifold with vanishing C-Bochner curvature tensor as mentioned in Lemma 1. At first let us calculate $B_k{}^D C^C C^B B_h{}^A K_{DCBA}$. From (5.1) and (5.11) we get

$$2(m+3) B_k{}^D C^C C^B B_h{}^A K_{DCBA} = B_k{}^D B_h{}^A K_{DA} + (g_{kh} - p_k p_h) K(C, C) - (k-4) g_{kh} + k p_k p_h. \tag{5.13}$$

Substituting (5.13) into (5.9), we obtain

$$B_k{}^D B_h{}^A K_{DA} = A_{kh}, \tag{5.14}$$

where we have put

$$A_{kh} = \frac{1}{m+2} \{(m+3)(K_{kh} + 2H_{kh}) + (g_{kh} - p_k p_h) K(C, C) - (k-4) g_{kh} + k p_k p_h\}. \tag{5.15}$$

Transvecting (5.10) with $B_j{}^C$ we get

$$B_i{}^A B_j{}^C S_{AC} = f_i{}^t A_{tj}. \tag{5.16}$$

Moreover, transvecting (5.9) with g^{kh} and using (5.8), we find

$$\tilde{K} = K + 2(C^D D^C D^B C^A K_{DCBA} - 2K(C, C) - H^2), \tag{5.17}$$

where $H^2 = h_{ji} h^{ji}$ and $\tilde{K} = g^{kh} \tilde{K}_{kh}$.

On the other hand a straightforward computation by using (5.8) gives

$$(m+3) C^D D^C D^B C^A K_{DCBA} = 4K(C, C) - (2k + 3m + 1), \tag{5.18}$$

Therefore substituting (5.18) into (5.17) yields

$$(m+4) K = \frac{(m+2)(m+3)}{m+1} \tilde{K} + 4(m+2) K(C, C) + \frac{2(m+2)(m+3)}{m+1} H^2 + \frac{6m^2 + 18m + 8}{m+1}. \tag{5.19}$$

Next we compute $B_k{}^D{}_j{}^C{}_h{}^B{}_i{}^A K_{DCBA}$. Using (5.1), (5.14) and (5.16) we have

$$(5.20)\quad B_k{}^D{}_j{}^C{}_i{}^B{}_h{}^A K_{DCBA} = -\frac{1}{2(m+3)}(A_{ki}g_{jh} - A_{ji}g_{kh} + g_{ki}A_{jh} - g_{ji}A_{kh}$$
$$+ f_{ki}f_{jh}A_{ti} - f_j{}^t f_{kh}A_t{}^i + f_{ki}f_j{}^t A_{th} - f_{ji}f_k{}^t A_{th} + 2f_k{}^t f_{ih}A_{tj} + 2f_{kj}f_i{}^t A_{th}$$
$$- A_{ki}p_jp_h + A_{ji}p_kp_h - p_kp_iA_{jh} + p_jp_iA_{kh})$$
$$+ \frac{k+2(m+1)}{2(m+3)}(f_{ki}f_{jh} - f_{ji}f_{kh} + 2f_{kj}f_{ih}) + \frac{k-4}{2(m+3)}(g_{ki}g_{jh} - g_{ji}g_{kh})$$
$$- \frac{k}{2(m+3)}(g_{ki}p_jp_h + p_kp_ig_{jh} - g_{ji}p_kp_h - p_jp_ig_{kh}).$$

On the other, using (5.2) and (5.19), we can easily verify the following identities:

$$(5.21)\quad \frac{k+2(m+1)}{2(m+3)} - \frac{K(C,C)}{(m+2)(m+3)} + \frac{k-4}{(m+2)(m+3)}$$
$$= \frac{\tilde{k}+2m}{2(m+2)} + \frac{H^2}{2(m+1)(m+2)},$$

$$(5.22)\quad \frac{k-4}{2(m+3)} - \frac{K(C,C)}{(m+2)(m+3)} + \frac{k-4}{(m+2)(m+3)}$$
$$= \frac{\tilde{k}-4}{2(m+2)} + \frac{H^2}{2(m+1)(m+2)},$$

$$(5.23)\quad \frac{\tilde{k}}{2(m+3)} - \frac{K(C,C)}{(m+2)(m+3)} + \frac{k-2}{(m+2)(m+3)}$$
$$= \frac{\tilde{k}}{2(m+2)} + \frac{H^2}{2(m+1)(m+2)},$$

where $\tilde{k} = \dfrac{\tilde{K}+2m}{2(m+1)}$.

Substituting (5.15) in (5.20) and making use of (5.21), (5.22) and (5.23), we obtain

$$(5.24)\quad B_k{}^D{}_j{}^C{}_i{}^B{}_h{}^A K_{DCBA} = -\frac{1}{2(m+2)}(\tilde{K}_{ki}g_{jh} - \tilde{K}_{ji}g_{kh} + g_{ki}\tilde{K}_{jh} - g_{ji}\tilde{K}_{kh} + S_{ki}f_{jh}$$
$$- S_{ji}f_{kh} + f_{ki}S_{jh} - f_{ji}S_{kh} + 2S_{kj}f_{ih} + 2f_{kj}S_{ih} - \tilde{K}_{ki}p_jp_h$$
$$+ \tilde{K}_{ji}p_kp_h - p_kp_i\tilde{K}_{jh} + p_jp_i\tilde{K}_{kh}) + \frac{\tilde{k}+2m}{2(m+2)}(f_{ki}f_{jh} -$$
$$f_{ji}f_{kh} + 2f_{kj}f_{ih}) + \frac{\tilde{k}-4}{2(m+2)}(g_{ki}g_{jh} - g_{ji}g_{kh})$$

$$-\frac{\tilde{k}}{2(m+2)}(g_{ki}p_jp_h+p_kp_ig_{jh}-g_{ji}p_kp_h-p_jp_ig_{kh})$$
$$+\frac{H^2}{2(m+1)(m+2)}(f_{ki}f_{jh}-f_{ji}f_{kh}$$
$$+2f_{kj}f_{ih}+g_{ki}g_{jh}-g_{ji}g_{kh}+g_{ji}p_kp_h+p_jp_ig_{kh}-g_{ki}p_jp_h$$
$$-p_kp_ig_{jh})+\frac{1}{m+2}(H_{ji}g_{kh}-H_{ki}g_{jh}+g_{ji}H_{kh}-g_{ki}H_{jh}$$
$$+Q_{ki}f_{jh}-Q_{ji}f_{kh}+f_{ki}Q_{jh}-f_{ji}Q_{kh}+2Q_{kj}f_{ih}+2f_{kj}Q_{ih}$$
$$+H_{ki}p_jp_h-H_{ji}p_kp_h+p_kp_iH_{jh}-p_jp_iH_{kh}),$$

where $H_{kh}=h_{ki}h_h^{\ i}$, $Q_{kh}=k_{kt}h_h^{\ t}$ and $S_{kh}=f_k^{\ t}\tilde{K}_{th}$, which and (5.3) imply

$$(5.25)\quad B_{kjih}=h_{kh}h_{ji}-h_{jh}h_{ki}+k_{kh}k_{ji}-k_{jh}k_{ki}+\frac{H^2}{2(m+1)(m+2)}(f_{ki}f_{jh}-f_{ji}f_{kh}$$
$$+2f_{kj}f_{ih}+g_{ki}g_{jh}-g_{ji}g_{kh}+g_{ji}p_kp_h+p_jp_ig_{kh}-g_{ki}p_jp_h-p_kp_ig_{jh})$$
$$+\frac{1}{m+2}(H_{ji}g_{kh}-H_{ki}g_{jh}+g_{ji}H_{kh}-g_{ki}H_{jh}+Q_{ki}f_{jh}-Q_{ji}f_{kh}+f_{ki}Q_{jh}$$
$$-f_{ji}Q_{kh}+2Q_{kj}f_{ih}+2f_{kj}Q_{ih}+H_{ki}p_jp_h-H_{ji}p_kp_h+p_kp_iH_{jh}-p_jp_iH_{kh}).$$

We transvect $h^{ki}g^{jh}$ to (5.25) and use $B_{kjik}=0$. Then we can easily see that

$$\frac{H^2}{(m+1)(m+2)}=0$$

with the help of (2.3), (4.21), (4.23), (4.24) and (4.25). Hence $H^2=0$, i.e, $h_{ji}h^{ji}=0$ and consequently $h_{ji}=0$. Since $k_{ji}=-h_{jt}f_i^{\ t}$, $h_{ji}=0$ implies $k_{ji}=0$. Therefore the Sasakian submanifold M^{2m+1} is totally geodesic. Thus we have Theorem 3.

By means of Theorem 3 and (5.25) we have $B_{kjih}=0$, which means that the C-Bochner curvature tensor of the Sasakian submanifold M^{2m+1} vanishes. Hence we have Corollary 4.

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