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$\mathcal{A}$-SURFACES WITH FLAT NORMAL CONNECTION

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1. Introduction.

Let M be an n-dimensional submanifold in an $(n+m)$-dimensional Riemannian manifold $\tilde{M}$. Let $\tilde{\nabla}$ and ∇ be the covariant differentiations of $\tilde{M}$ and M respectively. Let X and Y be arbitrary tangent vector fields on M. Then the second fundamental form h is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y).$$

$h(X, Y)$ is a normal vector field on M and is symmetric on X and Y. Let ξ be a normal vector field on M. We write

$$\tilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi,$$

where $-A_\xi(X)$ and $D_X\xi$ denote respectively the tangential and normal components of $\tilde{\nabla}_X\xi$. Then

$$\langle A_\xi(X), Y \rangle = \langle h(X, Y), \xi \rangle,$$

where $\langle\ ,\ \rangle$ denotes scalar product in $\tilde{M}$. A_ξ is called the *second fundamental tensor w. r. t.* ξ and D is called the *normal connection* of M in $\tilde{M}$.

Let $\xi_1, \xi_2, \cdots, \xi_m$ be an orthonormal basis for the normal space of M in $\tilde{M}$. Then *the mean curvature vector* H is given by

$$H = \frac{1}{n} \sum_{i=1}^{m} (Tr\ A_{\xi_x}) \xi_x.$$

If there exists a function λ on M such that

$$\langle h(X, Y), H \rangle = \lambda \langle X, Y \rangle,$$

then M is called a *pseudo-umbilical submanifold*. If the mean curvature vector H vanishes identically, then M is called a *minimal submanifold*.

Let ϕ be a normal vector field on M in $\tilde{M}$. Choose mutually orthogonal unit normal vector fields $\phi_1, \phi_2, \cdots, \phi_m$ on M such that

$$\phi = |\phi| \phi_1, \qquad |\phi| = \langle \phi, \phi \rangle^{1/2}$$

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Then the normal vector field

$$a(\phi)=\frac{|\phi|}{n}\sum_{y=2}^{m}[Tr(A_{\phi_1}\ A_{\phi_y})]\ \phi_y$$

is called the allied vector field of ϕ. In particular, $a(H)$ is called the *allied mean curvature vector field* of M. If the allied mean curvaturevector $a(H)$ vanishes identically, then M is called an *$\mathcal{A}$-submanifold.* ([2], [3]) From the definition it is easy to see that minimal submanifolds and pseudo-umbilical submanifolds are $\mathcal{A}$-submanifolds. Also it is clear that submanifolds whose *first normal space* N has dimension ≤ 1, in particular hypersurfaces, are $\mathcal{A}$-submanifolds (N is the orthogonal complement of $\{\xi | A_\xi=0\}$ in the normal space of M in $\tilde{M}$). There exist $\mathcal{A}$-submanifolds which do not belong to the classes of submanifolds mentioned above. In [2] examples of such $\mathcal{A}$-submanifolds are given: e.g. a certain 4-dimensional submanifold of the Euclidean 7-sphere $S^7(\sqrt{2})$ with radius $\sqrt{2}$, and the product submanifold $M\times M$ whereby M is a hypersurface of a Riemannian manifold such that the mean curvature vector of M is nowhere zero and M has second fundamental form with constant length. Regarding product submanifolds the following general result holds.

THEOREM A (B.-y. Chen [2]). *Let $M_i(i=1,2)$ be n_i-dimensional submanifolds of (n_i+m_i)-dimensional Riemannian manifolds $\tilde{M}_i$ with nowhere zero mean curvature vector H_i. Then the product manifold $M_1\times M_2$ is an $\mathcal{A}$-submanifold of $\tilde{M}_1\times\tilde{M}_2$ if and only if M_1 and M_2 are $\mathcal{A}$-submanifolds of $\tilde{M}_1$ and $\tilde{M}_2$ respectively and the second fundamental tensors of M_i in $\tilde{M}_i$ w.r.t. $\eta_i=\frac{H_i}{|H_i|}$ satisfy $Tr(A_{\eta_1}{}^2)=Tr(A_{\eta_2}{}^2)$.*

A submanifold M of Euclidean space E^{k+1} which is contained in a hypersphere S^k of E^{k+1} centered at the origin is called a spherical submanifold in E^{k+1}. Among all spherical submanifolds in Euclidean space, the $\mathcal{A}$-submanifolds are characterized by the following.

THEOREM B (C.-s. Houh [6]). *A spherical submanifold M in E^{k+1} is an $\mathcal{A}$-submanifold of E^{k+1} if and only if M is pseudo-umbilical in S^k.*

The curvature tensor K^N associated with the normal connection D,

$$K^N(X,Y)=[D_X,D_Y]-D_{[X,Y]},$$

is called the normal curvature tensor of M. If K^N vanishes identically, then M is said to have *flat normal connection*. It is well-known that the normal connection of M in $\tilde{M}$ is flat if and only if there exist locally mutually orthogonal unit normal vector fields ξ_x, $x\in\{1,2,\cdots,m\}$, which are parallel in the normal bundle ($D_X\xi_x=0$). The following results give characterizations for the flatness of the normal connection for submanifolds of conformally flat spaces.

THEOREM C (B.-y. Chen [4]). *A submanifold M has flat normal connection in a conformally flat space* $\tilde{M}$ *if and only if all second fundamental tensors* A_ξ *are simultaneously diagonalizable.*

THEOREM D (B.-y. Chen & L. Verstraelen [5]). *A surface M has flat normal connection in a conformally flat space* $\tilde{M}$ *if and only if M is umbilical w.r.t. an* $(m-1)$*-dimensional normal subbundle.*

An immediate consequence of Theorems C and D is given in the following.

LEMMA. *Let M be a surface in a conformally flat space* $\tilde{M}$*. If the normal connection of M in* $\tilde{M}$ *is flat, then dim* $N\leqq 2$.

The main purpose of this note is to show that among all surfaces with flat normal connection in a conformally flat space, the only $\mathcal{A}$-surfaces are the "trivial" ones. More precisely, we'll prove the following.

THEOREM. *Let M be a surface with flat normal connection in a conformally flat space* $\tilde{M}$*. Then M is an* $\mathcal{A}$*-surface if and only if M is minimal or pseudo-umbilical or dim* $N\leqq 1$.

As an application, we'll extend to $\mathcal{A}$-surfaces the following result regarding pseudo-umbilical surfaces.

THEOREM E (B.-y. Chen [1]). *Let M be a pseudo-umbilical surface with flat normal connection in a space form* $\tilde{M}$*. If M has constant mean curvature and constant Gauss curvature, then M is either flat or M is a totally umbilical submanifold of* $\tilde{M}$.

2. Preliminaries.

Throughout this paper we agree on the following ranges of indices: $A, B, C\in\{1, 2, \cdots, 2+m\}$; $i, j\in\{1, 2\}$; $r, s\in\{3, 4, \cdots, 2+m\}$; $\bar{r}, \bar{s}\in\{4, 5, \cdots, 2+m\}$.

Let M be a surface in a $(2+m)$-dimensional Riemannian manifold $\tilde{M}$. We choose a local field of orthonormal frames $b=\{e_1, e_2, e_3, \cdots, e_{2+m}\}$ in $\tilde{M}$ such that, restricted to M, the vectors e_1 and e_2 are tangent to M (and, consequently, the vectors $e_3, \cdots, e_{2+m}$ are normal to M). Let $\{\omega^1, \omega^2, \omega^3, \cdots, \omega^{2+m}\}$ be the field of dual frames w.r.t. the frame field choosen above. Then the structure equations of $\tilde{M}$ are given by

$$(1)\qquad d\omega^A=\sum_B \omega^B\wedge\omega_B{}^A, \qquad \omega_B{}^A+\omega_A{}^B=0,$$

(2) $$d\omega_A{}^B = \sum_C \omega_A{}^C \wedge \omega_C{}^B + \Omega_A{}^B,$$

whereby $\Omega_A{}^B$ are curvature 2-forms of $\tilde{M}$. In particular, when $\tilde{M}$ is a space of constant sectional curvature c,

(3) $$\Omega_A{}^B = c\omega^B \wedge \omega^A.$$

Restricting these forms to M, we have

(4) $$\omega^r = 0.$$

By taking exterior differentiation of (4) and using Cartan's lemma, we may write

(5) $$\omega_i{}^r = \sum_j h_{ij}{}^r \omega^j, \quad h_{ij}{}^r = h_{ji}{}^r.$$

The second fundamental form h, the second fundamental tensor A_r w. r. t. e_r and the mean curvature vector H of M in $\tilde{M}$ are respectively given by

(6) $$h = \sum_{r,i,j} h_{ij}{}^r \omega^i \omega^j e_r,$$

(7) $$A_r = [h_{ij}{}^r],$$

(8) $$H = \frac{1}{2} \sum_r (h_{11}{}^r + h_{22}{}^r) e_r.$$

3. Proof of Theorem.

Let M be a surface with flat normal connection in a conformally flat space $\tilde{M}$. Then, by Theorem C, we can choose the vectors e_1 and e_2 such that all second fundamental tensors A_r are diagonal, i. e. such that

(9) $$-h_{12}{}^r = 0.$$

Moreover, by Theorem D, we can choose the vectors e_r such that $m-1$ of them determine umbilical normal directions on M, i. e. such that

(10) $$A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad A_{\bar{r}} = \gamma_{\bar{r}} I,$$

whereby I denotes the identity. The second fundamental tensors w. r. t.

(11) $$\xi = e_{\bar{r}} \cos\theta + e_{\bar{s}} \sin\theta, \qquad \xi^{\perp} = -e_{\bar{r}} \sin\theta + e_{\bar{s}} \cos\theta$$

are

(12) $$A_\xi = (\gamma_{\bar{r}} \cos\theta + \gamma_{\bar{s}} \sin\theta) I, \qquad A_{\perp}{}^{\zeta} = (-\gamma_{\bar{r}} \sin\theta + \gamma_{\bar{s}} \cos\theta) I.$$

Thus, by an appropriate rotation (11) of $e_{\bar{r}}$ and $e_{\bar{s}}$, we can obtain orthonormal vectors ξ and $\xi^{\perp}$ such that

(13) $$A_\xi=\gamma_\xi I,\quad A_\xi^\perp=0.$$

Then it is clear that, by successive operations of this type, we can choose a local field of orthonormal frames $b=\{e_1, e_2, e_3, \cdots, e_{2+m}\}$ such that the corresponding second fundamental forms are given by

(14) $$A_3=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix},\quad A_4=\gamma I,\quad A_5=\cdots=A_{2+m}=0.$$

(This proves the Lemma).

W.r.t. such frames, the mean curvature field of M is given by

(15) $$H=\frac{\alpha+\beta}{2}e_3+\gamma e_4.$$

Let M be an A-surface in $\tilde{M}$. In case the mean curvature

(16) $$\mu=|H|=\left[\frac{(\alpha+\beta)^2}{4}+\gamma^2\right]^{1/2}$$

vanishes, then M is minimal and the Theorem holds. In case $\mu\neq 0$, we put

(17) $$\eta=\frac{H}{|H|}=\frac{1}{\mu}\left[\frac{\alpha+\beta}{2}e_3+\gamma e_4\right],\quad \eta^\perp=\frac{1}{\mu}\left[-\gamma e_3+\frac{\alpha+\beta}{2}e_4\right].$$

The second fundamental tensors of M w.r.t. the local field of orthonormal frames $b=\{e_1, e_2, \eta, \eta^\perp, e_5, \cdots, e_{2+m}\}$ are

(18) $$A_\eta=\frac{1}{2\mu}\begin{pmatrix}\alpha(\alpha+\beta)+2\gamma^2 & 0\\ 0 & \beta(\alpha+\beta)+2\gamma^2\end{pmatrix},\quad A_{\eta^\perp}=\frac{\gamma(\beta-\alpha)}{2\mu}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$
$$A_5=\cdots=A_{2+m}=0.$$

Thus, by definition, the allied mean curvature field $a(H)$ of M is given by

(19) $$a(H)=\frac{\mu}{2}[Tr(A_\eta A_\eta{}^\perp)]\eta^\perp.$$

From (18) we have

(20) $$A_\eta A_{\eta^\perp}=\frac{\gamma(\beta-\alpha)}{4\mu^2}\begin{pmatrix}\alpha(\alpha+\beta)+2\gamma^2 & 0\\ 0 & -\beta(\alpha+\beta)-2\gamma^2\end{pmatrix},$$

(21) $$Tr(A_\eta A_{\eta^\perp})=-\frac{\gamma(\alpha-\beta)^2(\alpha+\beta)}{4\mu^2}.$$

Consequently

(22) $$a(H)=-\frac{\gamma(\alpha-\beta)^2(\alpha+\beta)}{8\mu}\eta^\perp.$$

If $\gamma=0$ or $\alpha=\beta$, then we see from (18) that M is geodesic w. r. t. the $(m-1)$-dimensional normal subbundle spanned by $\eta^{\perp}, e_5, \cdots, e_{2+m}$, i. e. that H spans the first normal space N of M. If $\alpha+\beta=0$, then we see from (14) and (15) that M is umbilical w. r. t. $H(=\gamma e_4)$, i. e. that M is pseudo-umbilical. This completes the proof of the Theorem.

4. An application.

Let M be a surface with dim $N\leqq 1$ in a space form $\tilde{M}$ of curvature c. Then M has flat normal connection and w. r. t. a suitable choosen local field of orthonormal frames b we have

(23) $$A_3=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix},\quad A_r=0,$$

or equivalently

(24) $$\omega_1{}^3=\alpha\omega^1,\quad \omega_2{}^3=\beta\omega^2,\quad \omega_i{}^r=0.$$

From (1), (2), (3) and (24) it follows that

(25) $$d\omega^i=\omega^j\wedge\omega_j{}^i,$$

(26) $$d\omega_1{}^2=(c+\alpha\beta)\omega^2\wedge\omega^1,$$

(27) $$d\omega_i{}^3=\omega_i{}^j\wedge\omega_j{}^3.$$

By exterior differentiation of the first two equations of (24) we find

(28) $$d\alpha\wedge\omega^1=(\beta-\alpha)d\omega^1,\quad d\beta\wedge\omega^2=(\alpha-\beta)d\omega^2.$$

From (23) and (26) the mean curvature μ and the Gauss curvature G of M are respectively seen to be

(29) $$\mu=\frac{\alpha+\beta}{2},\quad G=c+\alpha\beta.$$

Suppose μ and G are both constant. Then, of course, α and β are both constant, and (28) implies that (i) $\alpha=\beta$ or (ii) $d\omega^1=d\omega^2=0$. In case (i) holds, then M is totally umbilical in $\tilde{M}$. If (ii) holds, then (25) shows that $\omega_1{}^2=0$, which in its turn shows that actually $G=0$.

Together with Theorem E and the previous Theorem, we thus obtain the following.

COROLLARY. *Let M be an $\mathcal{A}$-surface with flat normal connection in a space form $\tilde{M}$. If both the Gauss curvature and the mean curvature of M are constant, then M is flat or M is totally umbilical in $\tilde{M}$.*

Bibliography

[1] B.-y. Chen, *Pseudo-umbilical surfaces with constant Gauss curvature*, Proc. Edinburgh Math. Soc., **18** (2), 1972.

[2] B.-y. Chen, *Pseudo-umbilical submanifolds of a Riemannian manifold of constant curvature*, II, Journal Math. Soc. Japan, **25** (1), 1973.

[3] B.-y. Chen, *Geometry of submanifolds*, Marcel Dekker, New York, 1973.

[4] B.-y. Chen, *Some conformal invariants of submanifolds and their applications*, Boll. U.M.I., **4** (10), 1974.

[5] B.-y. Chen and L. Verstraelen, *Surfaces with flat normal connection*, Accad. Naz. Lincei, **LIX** (5), 1975.

[6] C.-s. Houch, *On spherical A-submanifolds*, Chinese Journal of Math., **2**(1), 1974.

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