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A STUDY ON EXTENSIONS OF TOPOLOGICAL SPACES

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0. Introduction.

The categorical approaches to the theory of extensions of topological spaces have been considered by several authors. M. Hušek [14] has defined the category **Ext** and constructed a basic functor G from the category of generalized proximity spaces into **Ext**. H. L. Bentley [3] has defined the category **Ex**, which has **Ext** as an isomorphism closed and epireflective subcategory, and investigated the properties of those categories. Moreover, H. Herrlich & H. L. Bentley [4] have shown that the correspondance between equivalence classes (modulo the relation of isomorphism) of strict extensions and concrete nearness spaces is one-one and onto.

On the top of these, we develop further theory on extensions of topological spaces. In particular, using the categorical languages, we try to find relationships between various extensions. The basic categorical structures of **Ext**, especially, those of the category **H-Ext** of Hausdorff extensions shall be investigated. Finally, we will show that, if **A** is an epireflective subcategory of **Haus**, then **A-Ext** is also an epireflective subcategory of **H-Ext**.

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1. The Category Ext.

The following definition is due to M. Hušek [14].

DEFINITION 1. The category **Ext** is defined as follow:

Objects of **Ext** are extensions $e : X \longrightarrow Y$, denoted by (e, X, Y); morphisms of (e, X, Y) to (e', X', Y') are all pairs (f, g) of continuous maps such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[e']{} & Y' \end{array}$$

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commutes.

If $(f, g) : (e, X, Y) \longrightarrow (e', X', Y')$ and $(f', g') : (e', X' Y') \longrightarrow (e'', X'', Y'')$ are morphisms of **Ext**, the composition of (f, g) and (f', g') is defined in an obvious way, i.e., $(f', g') \cdot (f, g) = (f' \cdot f, g' \cdot g)$.

The full subcategory of **Ext**, formed by Hausdorff (**A**, resp.) extensions of topological spaces will be denoted by **H-Ext** (**A-Ext**, resp.), where **A** is a subcategory of **Haus** and **A**-extension means its extension space belongs to **A**.

Since the composition of initial maps is again initial and homeomorphisms are initial, it is obvious that for a map e from a topological space X into a topological space Y, (e, X, Y) is an object of **Ext** if and only if e is injective, initial and dense.

THEOREM 2. *The category* **Ext** *has products.*

Proof. Let $(e_i, X_i, Y_i)_{i \in I}$ be any family of objects in **Ext** and let $\prod X_i$ and $\prod Y_i$ be the product spaces of $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$, respectively. Then clearly, $\prod e_i$ is injective and initial since e_j for each $j \in I$ and the projections $(p_j)_{j \in I}$ of $\prod X_i$ are both initial. $\prod e_i$ is obviousely dense. Thus $(\prod e_i, \prod X_i, \prod Y_i)$ is an object of **Ext**. We assert that $(\prod e_i, \prod X_i, \prod Y_i)$ is precisely the product of $(e_i, X_i, Y_i)_{i \in I}$ in **Ext**. Indeed, for each family of $(h_j, k_j) : (e, U, V) \longrightarrow (e_j, X_j, Y_j)\ (j \in I)$, $(h, k) : (e, U, V) \longrightarrow (\prod e_i, \prod X_i, \prod Y_i)$, where $h = \prod h_i$ and $k = \prod k_i$ is the unique morphism with $(p_j, q_j) \cdot (h, k) = (h_j, k_j)$. It follows from the definitions of h and k and the fact that $(q_j)_{j \in I}$ is a mono-source.

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[d, dashed, "h"] \arrow[dd, bend right=60, "h_j"'] & V \arrow[d, dashed, "k"'] \arrow[dd, bend left=60, "k_j"] \\ \prod X_i \arrow[r, "\prod e_i"] \arrow[d] & \prod Y_i \arrow[d, "q_j"] \\ X_j \arrow[r, "e_j"'] & Y_j \end{tikzcd}$$

Since **Haus** is productive, the following is immediate from the above Theorem.

THEOREM 3. *The category* **H-Ext** *has products.*

THEOREM 4. *The category* **Ext** *has equalizers.*

Proof. Let (f_1, g_1) and (f_2, g_2) be any pair of morphisms of **Ext** from (e_1, X_1, Y_1) to (e_2, X_2, Y_2) and let $(Z, h)=\mathrm{equ}(f_1, f_2)$, $(T, n)=\mathrm{equ}(g_1, g_2)$ in **Top**. Since $g_1\cdot(e_1\cdot h)=g_2\cdot(e_1\cdot h)$ and $(T, n)=\mathrm{equ}(g_1, g_2)$, there exists a unique continuous map $e' : Z\longrightarrow T$ such that $n\cdot e'=e_1\cdot h$. Let $e : Z\longrightarrow \mathrm{cl}(e'(Z))$ be the corestriction of e' to $\mathrm{cl}(e'(Z))$ and let m be the natural embedding of $\mathrm{cl}(e'(Z))$ into T. Then $e'=m\cdot e$; therfore e is injective and initial, and hence $(e, Z, \mathrm{cl}(e'(Z)))$ is an object of **Ext**. If we let $k=n\cdot m$, (h, k) is clearly a morphism of **Ext** with $(f_1, g_1)\cdot(h, k)=(f_2, g_2)\cdot(h, k)$. Assume that (u, v) is a morphism from (e'', U, V) to (e_1, X_1, Y_1) such that $(f_1, g_1)\cdot(u, v)=(f_2, g_2)\cdot(u, v)$. Then there is a pair (u', v') of continuous maps with $h\cdot u'=u$ and $n\cdot v'=v$, for $(Z, h)=\mathrm{equ}(f_1, f_2)$ and $(T, n)=\mathrm{equ}(g_1, g_2)$. Since n is a monomorphism, $e'\cdot u'=v'\cdot e''$.

$$
\begin{tikzcd}
U \arrow[rrrr, "e''"] \arrow[ddr, dashed, "u'"'] \arrow[dddr, bend right, "u"'] & & & & V \arrow[dll, "v''"'] \arrow[ddl, dashed, "v'"] \arrow[dddl, bend left, "v"] \\
 & & \mathrm{cl}(e'(Z)) \arrow[dr, "m"] \arrow[ddr, "k"] & & \\
 & Z \arrow[ur, "e"] \arrow[rr, "e'"] \arrow[d, "h"'] & & T \arrow[d, "n"] & \\
 & X_1 \arrow[rr, "e_1"] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] & & Y_1 \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "g_2"] & \\
 & X_2 \arrow[rr, "e_2"'] & & Y_2 &
\end{tikzcd}
$$

Noting $v'(V)\subset\mathrm{cl}(e'(Z))$, let v'' be the corestriction of v' to $\mathrm{cl}(e'(Z))$. Then it is easy to show that (u', v'') is the unique morphism from (e'', U, V) to $(e, Z, \mathrm{cl}(e'(Z)))$ with $(h, k)\cdot(u', v'')=(u, v)$. Hence $((e, Z, \mathrm{cl}(e'(Z))), (h, k))$ is the equalizer of (f_1, g_1) and (f_2, g_2).

Since **Haus** is hereditary, the following is immediate from the above Theorem.

THEOREM 5. *The category* **H-Ext** *has equalizers.*

Let U_1 (U_2, resp.) be a functor from **Ext** to **Top** that assigns to each object (e, X, Y) of **Ext**, the object X (Y, resp.) of **Top** and to any morphism (f, g) of **Ext**, the morphism f (g, resp.) of **Top**. Then the following is immediate from the above constructions.

THEOREM 6. (1) *The functor U_1 and U_2 preserve products.*

(2) *The functor U_1 preserves the equalizers but U_2 does not preserve the equalizers.*

For any category **C**, it is well-known [12] that the completeness of **C** is equivalent to the fact that **C** has products and equalizers. Thus we have, by Theorem 2 and 4, the following Theorem.

THEOREM 7. *The categories* **Ext** *and* **H-Ext** *are complete.*

Since U_1 preserves products and equalizers, U_1 preserves limits and the following is immediate.

PROPOSITION 8. *If a morphism (f, g) in* **Ext** (**H-Ext**, *resp.*) *is a monomorphism, then f is also a monomorphism in* **Top** (**Haus**, resp.).

THEOREM 9. *The category* **H-Ext** *of Hausdorff extensions is well-powered.*

Proof. Let (e, X, Y) be an arbitrary object of **H-Ext**. If $((e', X', Y'), (f, g))$ is a subobject of (e, X, Y), then Card $(X') \leqslant$ Card (X) since f is injective. Let $\mathfrak{S}$ be the class of Hausdorff spaces whose underlying sets are subsets of X and $\mathfrak{A}$ be the class of subobjects $((e', X', Y'), (f, g))$ of (e, X, Y) where $X' \in \mathfrak{S}$ and $Y' \subset P^2X$. Then it is easy to show that $\mathfrak{S}$ and $\mathfrak{A}$ are both sets. Now, we claim that $\mathfrak{A}$ is the representative class of subobjects of (e, X, Y). Indeed, if $((e'', U, V), (u, v))$ is a subobject of (e, X, Y), then there exists an $X' \in \mathfrak{S}$ and a homeomorphism $h : U \longrightarrow X'$ such that $u = f \cdot h$ where $f : X' \longrightarrow X$ is the inclusion map on the underlying sets. Since e'' : $U \longrightarrow V$ is a Hausdorff extension, we see that Card$(V) \leqslant$ Card$(P^2U) \leqslant$ Card (P^2X) and hence there exists an injective map $k : V \longrightarrow P^2X$. Let $Y' = k(V) \subset P^2X$ be the topological space with the topology transported by k. Then there exists $e' : X' \longrightarrow Y'$ such that (e', X', Y') is a member of $\mathfrak{A}$, and

$$\begin{tikzcd}
U \arrow[rrr, "e''"] \arrow[dr, "u"] \arrow[dd, "h"'] & & & V \arrow[dl, "v"'] \arrow[dd, "k"] \\
& X \arrow[r, "e"] & Y & \\
X' \arrow[ur, "f"] \arrow[rrr, "e'"'] & & & Y' \arrow[ul, "g"']
\end{tikzcd}$$

moreover, $(h, k) : (e'', U, V) \longrightarrow (e', X', Y')$ is an isomorphism.

It is well-known [11] that if a category $\mathbf{C}$ is complete and well-powered, then the following are true;

(1) $\mathbf{C}$ is an (epi, extremal mono) category,

(2) $\mathbf{C}$ is an (extremal epi, mono) category,

(3) $\mathbf{C}$ is uniquely (extremal epi, bimorphism, extremal mono)-factorizable. Using this facts together with Theorem 7 and 9, we have the following;

THEOREM 10. (1) **H-Ext** *is an (epi, extremal mono) category.*

(2) **H-Ext** *is an (extremal epi, mono) category.*

(3) **H-Ext** *is uniquely (extremal epi, bimorphism, extremal mono)-factorizable.*

PROPOSITION 11. *If* $(f, g) : (e, X, Y) \longrightarrow (e', X', Y')$ *is an epimorphism in* **Ext**, *then* g *is surjective.*

Proof. Suppose that g is not surjective, i.e., $g(Y) \neq Y'$ and let $Z = \{1, 2\}$ with the indiscrete topology. Define $v_1 : Y' \longrightarrow Z$, and $v_2 : Y' \longrightarrow Z$ as follow : $v_1(y) = 1$ for every $y \in Y'$ and

$$v_2(y) = \begin{cases} 1, & \text{if } y \in g(Y) \\ 2, & \text{otherwise.} \end{cases}$$

Define $u_1 : X' \longrightarrow Z$ and $u_2 : X' \longrightarrow Z$ as follow : $u_1(x) = 1$ for every $x \in X'$ and

$$u_2(x) = \begin{cases} 1, & \text{if } x \in e'^{-1}(g(Y)) \\ 2, & \text{otherwise.} \end{cases}$$

Then it is clear that (u_1, v_1) and (u_2, v_2) are morphisms from (e', X', Y') to $(1_Z, Z, Z)$ in **Ext** such that $(u_1, v_1) \cdot (f, g) = (u_2, v_2) \cdot (f, g)$ but $(u_1, v_1) \neq (u_2, v_2)$, and hence (f, g) is not an epimorphism.

Let F be a subset of a Hausdorff space X. It is well-known that F is closed if and only if there exist continuous maps f and g from X into a Hausdorff space Y such that (F, i) is the equalizer of f and g, where i is the natural embedding of F into X. It is also known that a morphism $f : X \longrightarrow Y$ in **Haus** is an epimorphism if and only if f is dense.

PROPOSITION 12. *A morphism* (f, g) *in* **H-Ext** *is an epimorphism if and only if* g *is dense.*

Proof. Let $(f, g) : (e, X, Y) \longrightarrow (e', X', Y')$ and suppose that g is not dense. Then there exist morphisms v_1 and v_2 in **Haus** such that $\mathrm{equ}(v_1, v_2) = \mathrm{cl}(g(Y))$ but $v_1 \neq v_2$. Define u_1 and u_2 by $u_1 = v_1 \cdot e'$ and $u_2 = v_2 \cdot e'$. Obviously, (u_1, v_1) and (u_2, v_2) are morphisms in **H-Ext** with $v_1 \cdot g = v_2 \cdot g$. For each $x \in X$, $u_1 \cdot f(x) = v_1 \cdot e' \cdot f(x) = v_1 \cdot g \cdot e(x) = v_2 \cdot g \cdot e(x) = v_2 \cdot e' \cdot f(x) = u_2 \cdot f(x)$ which shows that $u_1 \cdot f = u_2 \cdot f$. Thus $(u_1, v_1) \cdot (f, g) = (u_2, v_2) \cdot (f, g)$, therefore (f, g) is not an epimorphism. To prove the converse, suppose that (u_1, v_1) and (u_2, v_2) are morphisms from (e', X', Y') to (e'', X'', Y'') in **H-Ext** such that $(u_1, v_1) \cdot (f, g) = (u_2, v_2) \cdot (f, g)$, i.e., $u_1 \cdot f = u_2 \cdot f$ and $v_1 \cdot g = v_2 \cdot g$. By hypothesis, $v_1 \cdot g = v_2 \cdot g$ implies $v_1 = v_2$. Hence $e'' \cdot u_1 = v_1 \cdot e' = v_2 \cdot e' = e'' \cdot u_2$. Since e'' is a monomorphism, we have $u_1 = u_2$. Thus $(u_1, v_1) = (u_2, v_2)$.

THEOREM 13. *The category* **H-Ext** *is co-well-powered.*

Proof. Let (e, X, Y) be any object of **H-Ext** and $\mathfrak{S}$ be the set of Hausdorff spaces whose underlying sets are subsets of $P^2 Y$. Then by the same arguments as those in Theorem 9, the class $\mathfrak{A}$ of quotient objects $((f, g), (e', X', Y'))$ of (e, X, Y) such that $Y' \in \mathfrak{S}$ and the underlying set of X' is a subsets of $P^2 Y$, is the representative class of the quotient objects of (e, X, Y) which is a set.

THEOREM 14. *If* **A** *is an epireflective subcategory of* **Haus**, *then* **H-Ext** *is also epireflective in* **H-Ext**.

Proof. Since **H-Ext** is complete, well-powered and co-well-powered by Theorem 7, 9 and 13, it is enough to show that **A-Ext** is strongly closed under the formation of products and equalizers in **H-Ext**. For a family $(e_i, X_i, Y_i)_{i \in I}$ of objects in **A-Ext**, the product of the family is given by $(\prod e_i, \prod X_i, \prod Y_i)$, which belongs to **A-Ext**, for **A** is productive. For a pair of morphisms (f_1, g_1) and (f_2, g_2) from (e_1, X_1, Y_1) to (e_2, X_2, Y_2) in **H-Ext** such that (e_1, X_1, Y_1) belongs to **A-Ext** and for the equalizer (e, X, Y) of the pair in **H-Ext**, Y is a closed subspace of Y_1 by Theorem 4. Since **A** is again closed hereditary, (e, X, Y) also belongs to **A-Ext**. This completes the proof.

We recall that a subcategory **A** of **Haus** is epireflective in **Haus** if and only if it is both productive and closed hereditary. Examples of epireflective subcategories in **Haus** are:

(1) the category of all compact spaces and continuous maps [6], [15],

(2) the category of real compact spaces and continuous maps [13],

(3) the category of zero-dimensional compact spaces and continuous maps [1],

(4) the category of E-compact spaces for a Hausdorff space E and continuous maps [7],

(5) the category of $\mathfrak{E}$-compact spaces for a class $\mathfrak{E}$ of Hausdorff spaces and continuous maps [8],

(6) the category of k-compact spaces for an infinite cardinal k and continuous maps [9],

(7) the category of m-ultra compact spaces for an infinite cardinal m and continuous maps [17],

(8) the category of completely regular spaces and continuous maps [6],

(9) the category of T_3-spaces and continuous maps [16].

Using the above Theorem, for any category **A** of (1)-(9), **A-Ext** is epireflective in **H-Ext.**

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