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REMARKS ON INFINITESIMAL CONFORMAL AND MINIMAL VARIATIONS OF A SUBMANIFOLD

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§ 0. Introduction.

Various infinitesimal variations (for examples; isometric, conformal, affine, affine collinear etc.) of submanifolds of a Riemannian manifold have been studied in [1]~[4].

An infinitesimal variation which carries a minimal submanifold into a minimal submanifold is said to be *minimal* ([2]). In the previous paper ([2]) the present authors studied infinitesimal isometric and minimal variations of a compact orientable submanifold.

In the present paper we investigate a conformal and minimal variation of a compact orientable submanifold of a Riemannian manifold (See Theorem 2.1). We remark the correction of the previous paper [2] (See § 2).

§ 1. Preliminaries. ([1], [4])

Let M^m be an m-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and denote by g_{ji}, Γ_{ji}^h, ∇_j, $K_{kji}{}^h$ and K_{ji} the metric tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to Γ_{ji}^h, the curvature tensor and the Ricci tensor of M^m respectively, where here and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, 3, \cdots, m\}$.

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and denote by g_{cb}, Γ_{cb}^a, ∇_c, $K_{dcb}{}^a$ and K_{cb} the corresponding quantities of M^n respectively, where here and in the sequel the indices $a, b, c, d, \cdots$ run over the range $\{1, 2, 3, \cdots, n\}$.

We suppose that M^n is isometrically immersed in M^m by the immersion $i: M^n \to M^m$ and identify $i(M^n)$ with M^n itself. We represent the immersion by $x^h = x^h(y^a)$ and put $B_b{}^h = \partial_b x^h (\partial_b = \partial/\partial y^b)$. Then $B_b{}^h$ are n linearly independent vectors of M^m tangent to M^n. Since the immersion i is isometric, we obtain

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(1.1) $$g_{cb}=g_{ji}B_c{}^jB_b{}^i.$$

We denote by $C_y{}^h$ $m-n$ mutually orthogonal unit normals to M^n, where here and in the sequel the indices x, y, z run over the range $\{n+1, n+2, \cdots, m\}$. Then the metric tensor of the normal bundle of M^n is given by

(1.2) $$g_{zy}=C_z{}^jC_y{}^ig_{ji}.$$

It is well known that $\Gamma_{cb}{}^a$ and $\Gamma_{ji}{}^h$ are related by

(1.3) $$\Gamma_{cb}{}^a=(\partial_cB_b{}^h+\Gamma_{ji}{}^hB_c{}^jB_b{}^i)B^a{}_h,$$

where $B^a{}_h=B_b{}^ig^{ba}g_{ih}$, $(g^{ba})=(g_{ba})^{-1}$, and the components $\Gamma_{cy}{}^x$ of the connection induced in the normal bundle are given by

(1.4) $$\Gamma_{cy}{}^x=(\partial_cC_y{}^h+\Gamma_{ji}{}^hB_c{}^jC_y{}^i)C^x{}_h,$$

where $C^x{}_h=C_y{}^ig^{yx}g_{ih}$, g^{yx} being contravariant components of the metric tensor g_{yx} of the normal bundle.

If we denote by $\nabla_cB_b{}^h$ and $\nabla_cC_y{}^h$ the van der Waerden-Bortolotti covariant derivatives of $B_b{}^h$ and $C_y{}^h$ along the M^n respectively, that is, if we put

(1.5) $$\nabla_cB_b{}^h=\partial_cB_b{}^h+\Gamma_{ji}{}^hB_c{}^jB_b{}^i-\Gamma_{cb}{}^aB_a{}^h$$

and

(1.6) $$\nabla_cC_y{}^h=\partial_cC_y{}^h+\Gamma_{ji}{}^hB_c{}^jC_y{}^i-\Gamma_{cy}{}^xC_x{}^h,$$

then we can write equations of Gauss and those of Weingarten in the form

(1.7) $$\nabla_cB_b{}^h=h_{cb}{}^xC_x{}^h,$$

(1.8) $$\nabla_cC_y{}^h=-h_c{}^a{}_yB_a{}^h$$

respectively, where $h_{cb}{}^x$ are the second fundamental tensor of M^n with respect to the normals $C_x{}^h$ and $h_c{}^a{}_x=h_{cbx}g^{ba}$.

Equations of Gauss and Codazzi are respectively

(1.9) $$K_{dcb}{}^a=K_{kji}{}^hB_d{}^kB_c{}^jB_b{}^iB^a{}_h+h_d{}^a{}_xh_{cb}{}^x-h_c{}^a{}_xh_{db}{}^x,$$

(1.10) $$0=K_{kji}{}^hB_d{}^kB_c{}^jB_b{}^iC^x{}_h-(\nabla_dh_{cb}{}^x-\nabla_ch_{db}{}^x).$$

From (1.9) we have

(1.11) $$K_{cb}=K_{kjih}B_c{}^kB^{ji}B_b{}^h+h_e{}^e{}_xh_{cb}{}^x-h_{ce}{}^xh_b{}^e{}_x,$$

where $B^{ji}=g^{cb}B_c{}^jB_b{}^i$ and $K_{kjih}=K_{kji}{}^tg_{th}$.

We now consider a variation of M^n in M^m given by $\bar{x}^h=x^h+f^h(y)\varepsilon$, where ε is an infinitesimal. We then have

(1.12) $$\bar{B}_b{}^h=B_b{}^h+(\partial_bf^h)\varepsilon,$$

where $\bar{B}_b{}^h = \partial_b \bar{x}^h$ are n linearly independent vectors tangent to the deformed submanifold at the deformed point $(\bar{x}^h)$. If we displace $B_b{}^h$ parallely from the point $(\bar{x}^h)$ to (x^h), we then obtain the vectors

$$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma_{ji}{}^h(x + f\varepsilon) f^j B_b{}^i \varepsilon,$$

at the point (x^h), or

$$\tilde{B}_b{}^h = B_b{}^h + (\nabla_b f^h)\varepsilon, \tag{1.13}$$

neglecting the terms of order higher than one with respect to ε, where

$$\nabla_b f^h = \partial_b f^h + \Gamma_{ji}{}^h B_b{}^j f^i.$$

In the sequel we always neglect terms of order higher than one with respect to ε. Thus putting

$$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h, \tag{1.14}$$

we have from (1.13)

$$\delta B_b{}^h = (\nabla_b f^h)\varepsilon. \tag{1.15}$$

If we put

$$f^h = f^a B_a{}^h + f^x C_x{}^h, \tag{1.16}$$

we have

$$\nabla_b f^h = (\nabla_b f^a - h_b{}^a{}_x f^x) B_a{}^h + (\nabla_b f^x + h_{ba}{}^x f^a) C_x{}^h \tag{1.17}$$

because of (1.7) and (1.8).

From (1.14), (1.15) and (1.17), we obtain

$$\tilde{B}_b{}^h = \{\delta_b^a + (\nabla_b f^a - h_b{}^a{}_x f^x)\varepsilon\} B_a{}^h + (\nabla_b f^x + h_{ba}{}^x f^a) C_x{}^h \varepsilon. \tag{1.18}$$

Now applying the operator δ to (1.1) and using (1.15), (1.17) and $\delta g_{ji} = 0$, we have

$$\delta g_{cb} = (\nabla_c f_b + \nabla_b f_c - 2h_{cbx} f^x)\varepsilon, \tag{1.19}$$

where $f_b = g_{ba} f^a$, from which

$$\delta g^{ba} = -(\nabla^b f^a + \nabla^a f^b - 2h^{ba}{}_x f^x)\varepsilon. \tag{1.20}$$

A variation of a submanifold for which $\delta g_{cb} = 0$ is said to be isometric and that for which δg_{cb} is proportional to g_{cb} is said to be conformal (cf. [4]).

Thus we have from (1.19) the following assertion:

In order for a variation of a sumanifold to be isometric or conformal, it is necessary and sufficient that

(1.21) $$\nabla_c f_b+\nabla_b f_c-2h_{cbx}f^x=0,$$

or

(1.22) $$\nabla_c f_b+\nabla_b f_c-2h_{cbx}f^x=2\lambda g_{cb}$$

respectively λ being a certain function.

We denote by $\bar{C}_y{}^h$ $m-n$ mutually orthogonal unit normals to the deformed submanifold and by $\tilde{C}_y{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by parallel displacement of $C_y{}^h$ from the point $(\bar{x}^h)$ to (x^h). Then we have

$$\tilde{C}_y{}^h=\bar{C}_y{}^h+\Gamma_{ji}^h(x+f\varepsilon)f^jC_y{}^i\varepsilon.$$

We put

(1.23) $$\delta C_y{}^h=\tilde{C}_y{}^h-C_y{}^h$$

and assume that $\delta C_y{}^h$ is of the form

(1.24) $$\delta C_y{}^h=(f_y{}^aB_a{}^h+f_y{}^xC_x{}^h)\varepsilon.$$

Applying the operator δ to $B_c{}^jC_y{}^ig_{ji}=0$ and using (1.15), (1.17), (1.24) and $\delta g_{ji}=0$, we obtain

$$(\nabla_b f_y+h_{bay}f^a)+f_{yb}=0,$$

where $f_{yb}=f_y{}^cg_{cb}$, or

(1.25) $$f_y{}^a=-(\nabla^a f_y+h_b{}^a{}_yf^b).$$

Applying also the operator δ to $C_x{}^jC_y{}^ig_{ji}=\delta_{xy}$ and using (1.24) and $\delta g_{ji}=0$, we have

(1.26) $$f_{yx}+f_{xy}=0,$$

where $f_{yx}=f_y{}^zg_{zx}$.

The variations of the second fundamental tensor $h_{cb}{}^x$ and Christoffel symbols are given by ([4])

(1.27) $$\delta h_{cb}{}^x=\{f^d\nabla_d h_{cb}{}^x+h_{eb}{}^x(\nabla_c f^e)+h_{ce}{}^x(\nabla_b f^e)-h_{cb}{}^yf_y{}^x\}\varepsilon$$
$$+\{\nabla_c\nabla_b f^x+K_{kji}{}^hC_y{}^kB_c{}^jB_b{}^iC^x{}_hf^y\}\varepsilon,$$

(1.28) $$\delta\Gamma_{cb}^a=(\nabla_c\nabla_b f^a+K_{dcb}{}^af^d)\varepsilon$$
$$-\{\nabla_c(h_{bex}f^x)+\nabla_b(h_{cex}f^x)-\nabla_e(h_{cbx}f^x)\}g^{ea}\varepsilon.$$

A variation of a submanifold for which $\delta\Gamma_{cb}^a=0$ is said to be affine.

§2. Integral formulas on minimal variations.

We now suppose that a variation of the submanifold M^n is conformal, we then have from (1.22)

$$(2.1)\qquad (\nabla_c f_b) h^{cb}{}_x f^x = (h_{cbx} f^x)(h^{cb}{}_y f^y) + \lambda h_b{}^b{}_x f^x,$$

$$(2.2)\qquad \nabla_b f^b = h_b{}^b{}_x f^x + \lambda n,$$

and

$$(2.3)\qquad \nabla_c \nabla_b f_a + \nabla_c \nabla_a f_b = 2\nabla_c (h_{bax} f^x + \lambda g_{ba}),$$

from which, using the Ricci-identity

$$(2.4)\qquad \nabla_c \nabla_b f_a + \nabla_a \nabla_c f_b - K_{cabd} f^d = 2\nabla_c (h_{bax} f^x + \lambda g_{ba}),$$

or, transvecting g^{cb} and taking account of (2.2)

$$(2.5)\qquad \nabla^c \nabla_c f_a + K_{ab} f^b - 2\nabla^e (h_{aex} f^x) + \nabla_a (h_b{}^b{}_x f^x) + (n-2)\nabla_a \lambda = 0.$$

From (1.11) and (2.5) we find

$$(2.6)\qquad \begin{aligned}(\nabla^c \nabla_c f_a) f^a = &- K_{kjih}(f^c B_c{}^k) B^{ji} (f^b B_b{}^h) - h_e{}^e{}_x (h_{cb}{}^x f^c f^b)\\ &+ (h_{ce}{}^x f^c)(h_b{}^e{}_x f^b) + 2 f^a \nabla^e (h_{aex} f^x)\\ &- f^a \nabla_a (h_b{}^b{}_x f^x) - (n-2) f^a \nabla_a \lambda.\end{aligned}$$

We now compute the variation of the mean curvature. For a variation of the submanifold, we have

$$\delta(g^{cb} h_{cb}{}^x) = (\delta g^{cb}) h_{cb}{}^x + g^{cb} \delta h_{cb}{}^x.$$

Substitution (1.22) and (1.27) yields

$$(2.7)\qquad \begin{aligned}\delta h_e{}^{ex} = [&\nabla^c \nabla_c f^x + K_{kjih} C_y{}^k B^{ji} C_x{}^h f^y + (h_{cb}{}^x)(h^{cb}{}_y f^y)\\ &+ f^c \nabla_c h_b{}^{bx} - h_a{}^{ay} f_y{}^x]\varepsilon.\end{aligned}$$

A variation carries a minimal submanifold into a minimal submanifold, that is, $\delta h_e{}^{ex} = 0$ and $h_e{}^{ex} = 0$, then we say that the variation is minimal ([2]).

For a minimal variation, we have from (2.7)

$$(2.8)\qquad (\nabla^c \nabla_c f^x) f_x = -K_{kjih}(f^y C_y{}^k) B^{ji} (f^x C_x{}^h) - (h_{cbx} f^x)(h^{cb}{}_y f^y).$$

For a conformal and minimal variation, (2.1), (2.2) and (2.6) reduce respectively to

$$(2.9)\qquad (\nabla_c f_b) h^{cb}{}_x f^x = (h_{cbx} f^x)(h^{cb}{}_y f_y),$$

$$(2.10)\qquad \nabla_b f^b = n\lambda$$

and

$$(\nabla^c\nabla_c f_a)f^a = -K_{kjih}(f^c B_c{}^k)B^{ji}(f^b B_b{}^h) - (n-2)f^a\nabla_a\lambda + (h_{ce}{}^x f^c)(h_b{}^e{}_x f^b) + 2f^a\nabla^e(h_{aex}f^x). \tag{2.11}$$

On the other hand, we have

$$\frac{1}{2}\Delta(f_a f^a + f_x f^x) = (\nabla^c\nabla_c f_a)f^a + (\nabla^c\nabla_c f_x)f^x + (\nabla_c f_b)(\nabla^c f^b) + (\nabla_c f_x)(\nabla^c f^x) \tag{2.12}$$

where $\Delta = g^{cb}\nabla_c\nabla_b$, and we get

$$\begin{aligned} K_{kjih}f^k B^{ji} f^h &= K_{kjih}(f^a B_a{}^k + f^x C_x{}^k)B^{ji}(f^b B_b{}^h + f^y C_y{}^h) \\ &= K_{kjih}(f^a B_a{}^k)B^{ji}(f^b B_b{}^h) + K_{kjih}(f^x C_x{}^k)B^{ji}(f^y C_y{}^h) \\ &\quad + 2f^a f^x(\nabla_a h_e{}^e{}_x - \nabla^e h_{aex}) \end{aligned} \tag{2.13}$$

with the help of (1.10) and (1.16).

Substituting (2.8), (2.11) and (2.13) with $h_e{}^e{}_x = 0$ into (2.12), we find

$$\begin{aligned} \frac{1}{2}\Delta(f_a f^a + f_x f^x) &= -K_{kjih}f^k B^{ji} f^h - 2f^a f^x(\nabla^e h_{eax}) \\ &\quad + (h_{ce}{}^x f^c)(h_b{}^e{}_x f^b) + 2f^a\nabla^e(h_{aex}f^x) \\ &\quad - (h_{cbx}f^x)(h^{cb}{}_y f^y) - (n-2)f^a\nabla_a\lambda + (\nabla_c f_b)(\nabla^c f^b) \\ &\quad + (\nabla_c f_x)(\nabla^c f^x) \\ &= -K_{kjih}f^k B^{ji} f^h - (h_{cbx}f^x)(h^{cb}{}_y f^y) \\ &\quad + (\nabla_c f_b)(\nabla^c f^b) + \|\nabla_c f_x + h_{cex}f^e\|^2 - (n-2)f^a\nabla_a\lambda. \end{aligned} \tag{2.14}$$

Since we have from (2.10)

$$\nabla^a(\lambda f_a) = (\nabla_a\lambda)f^a + n\lambda^2,$$

(2.14) reduces to

$$\nabla^c u_c = -K_{kjih}f^k B^{ji} f^h + \|\nabla_c f_b - h_{cb}{}^x f_x\|^2 + \|\nabla_c f_x + h_{cex}f^e\|^2 + n(n-2)\lambda^2 \tag{2.15}$$

with the help of (2.9), where we have put

$$u_c = \frac{1}{2}\nabla_c(f_a f^a + f_x f^x) + (n-2)\lambda f_c.$$

Applying Green's theorem on (2.15), we have

THEOREM 2.1. *Let M^n be a compact orientable submanifold of a Riemannian manifold. If an infinitesimal variation $\bar{x}^h = x^h + f^h\varepsilon$ of M^n is conformal (or* *isometric) and minimal, then we have*

$$\int_{M^n} K_{kjih} f^k B^{ji} f^h \geqq 0.$$

Moreover, if $K_{kjih} f^k B^{ji} f^h \leqq 0$, *then the variation is parallel.*

CORRECTION. The terms $K_{kjih} f^k B^{ji} f^h + 2(h_{cbx} f^x)(h^{cb}{}_y f^y)$ appeared in Lemma 3.2 and Theorem 4.1 of [2] are replaced by $K_{kjih} f^k B^{ji} f^h$.

Finally, we suppose that a variation of the submanifold M^n is affine. We then have from (1.28)

$$\nabla^c \nabla_c f_a + K_{ab} f^b - 2\nabla^e (h_{aex} f^x) + \nabla_a (h_b{}^b{}_x f^x) = 0. \tag{2.16}$$

So (2.6) with $\lambda=0$ is valid.

For an affine and minimal variation, we see that (2.8) and (2.11) with $\lambda=0$ are true.

Substituting (2.8), (2.11) with $\lambda=0$ and (2.13) with $h_e{}^e{}_x=0$ into (2.12), we find

$$\begin{aligned} &\frac{1}{2}\Delta(f_a f^a + f_x f^x) \\ &= -K_{kjih} f^k B^{ji} f^h - (h_{cbx} f^x)(h^{cb}{}_y f^y) \\ &\quad + \|\nabla_c f_b\|^2 + \|\nabla_c f_x + h_{cex} f^x\|^2. \end{aligned}$$

Thus we have

THEOREM 2.2. *Let M^n be a compact orientable submanifold of a Riemannian manifold. If an infinitesimal variation of M^n is affine and minimal, and if*

$$K_{kjih} f^k B^{ji} f^h + (h_{cbx} f^x)(h^{cb}{}_y f^y) \leqq 0,$$

then the variation is parallel.

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