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INFINITESIMAL VARIATIONS OF INVARIANT SUBMANIFOLDS WITH NORMAL (f, g, u, v, λ)-STRUCTURE

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Introduction.

Recently infinitesimal variations of submanifolds of a Riemannian manifold have been studied by Chen (cf. [2]), Yano (cf. [2], [7]) and many authors.

On the other hand Yano, Okumura and one of the present authors (cf. [9]) have studied infinitesimal variations of invariant submanifolds of a Kaehlerian manifold.

The purpose of the present paper is to study infinitesimal variations of invariant submanifolds with induced (f, g, u, v, λ)-structure of the ambient manifold with normal (f, g, u, v, λ)-structure.

In the preliminary § 1, we state some properties of invariant submanifolds with induced (f, g, u, v, λ)-structure of the ambient manifold with normal (f, g, u, v, λ)-structure.

In § 2, we prove fundamental formulas in the theory of infinitesimal variations and study invariance-preserving variations, that is, infinitesimal variations which carry an invariant submanifold into an invariant submanifold. In § 3, we study f-preserving variations, that is, invariance-preserving variations which preserves a tensor field f of type (1, 1) in (f, g, u, v, λ)-structures induced on invariant submanifolds. In § 4, we compute the variations of u, v, and λ.

In § 5, we calculate $T_{cb}T^{cb}$, T_{cb} being defined in such a way that the variation is f-preserving if and only if $T_{cb}=0$. In the later part of § 5, we consider an infinitesimal invariance-preserving variation of a compact invariant submanifold with induced normal (f, g, u, v, λ)-structure of the even dimensional sphere S^{2m} with normal (f, g, u, v, λ)-structure and get an integral formula involving $T_{cb}T^{cb}$.

In the last § 6, under the same assumptions as those in the later part of § 5, we prove theorems on f-preserving and isometric variations.

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§1. Invariant submanifolds with induced (f, g, u, v, λ)-structure of a manifold with normal (f, g, u, v, λ)-structure.

Let M^{2m} be a real $2m$-dimensional manifold covered by a system of coordinate neighbourhoods $\{U; x^h\}$, in which a manifold with a tensor field f of type $(1, 1)$, a Riemannian metric g, two 1-forms u, v and a function λ satisfying

$$(1.1)\qquad \begin{cases} f_j{}^t f_t{}^h = -\delta_j^h + u_j u^h + v_j v^h \\ f_j{}^s f_i{}^t g_{st} = g_{ji} - u_j u_i - v_j v_i, \\ u_t f_j{}^t = \lambda v_j, \quad f_t{}^h u^t = -\lambda v^h, \\ v_t f_j{}^t = -\lambda u_j, \quad f_t{}^h v^t = \lambda u^h, \\ u_t u^t = v_t v^t = 1 - \lambda^2, \quad u_t v^t = 0, \end{cases}$$

$f_i{}^h, g_{ji}, u_i, v_i$ and λ being respectively components of f, g, u, v and λ with respect to a local coordinate system, u^h and v^h being defined by $u_i = g_{ih} u^h$ and $v_i = g_{ih} v^h$ respectively, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, 3, \cdots, 2m\}$, then the structure is called an (f, g, u, v, λ)-structure (cf. [4], [8]). It is known that such a manifold is even dimensional (cf. [4]). If we put $f_{ji} = f_j{}^t g_{ti}$, we can easily see that f_{ji} is skew-symmetric.

We put

$$S_{ji}{}^h = [f, f]_{ji}{}^h + (\nabla_j u_i - \nabla_i u_j) u^h + (\nabla_j v_i - \nabla_i v_j) v^h,$$

$[f, f]_{ji}{}^h$ denoting the Nijenhuis tensor formed with $f_i{}^h$ and ∇_i the operator of covariant differentiation with respect to the Christoffel symbols Γ_{ji}^h formed with g_{ji}. If $S_{ji}{}^h$ vanishes, then the (f, g, u, v, λ)-structure is said to be normal (cf. [8]).

The following theorem is well known (cf. [8]).

THEOREM 1.1. *Let M^{2m} be a manifold with normal (f, g, u, v, λ)-structure satisfying $\nabla_j v_i - \nabla_i v_j = 2 f_{ji}$ (or equivalently $\nabla_j u_i + \nabla_i u_j = -2\lambda g_{ji}$). If the function $\lambda(1-\lambda^2)$ does not vanish almost everywhere, then we have*

$$(1.2)\qquad \begin{cases} \nabla_j f_i{}^h = g_{ji}(\phi u^h - v^h) - \delta_j^h (\phi u_i - v_i), \\ \nabla_j u_i = -\lambda g_{ji} - \phi f_{ji}, \quad \nabla_j v_i = -\phi \lambda g_{ji} + f_{ji}, \\ \nabla_j \lambda = u_j + \phi v_j, \end{cases}$$

ϕ being constant. Moreover, if M^{2m} is complete and dim $M^{2m}>2$, then M^{2m} is isometric with an even dimemsional sphere.

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods $\{V; y^a\}$ and with metric tensor g_{cb}, where here and in the sequel, the indices $a, b, c, \cdots$ run over the range $\{1, 2, 3, \cdots, n\}$. We assume that M^n is isometrically immersed in M^{2m} by the immersion i: $M^n \longrightarrow M^{2m}$ and identify $i(M^n)$ with M^n itself. We represent the immersion i locally by $x^h = x^h(y^a)$ and put $B_b{}^h = \partial_b x^h$, $\partial_b = \partial/\partial y^b$, which are n linearly independent vectors of M^{2m} tangent to M^n. Since the immersion i is isometric, we have

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i. \tag{1.3}$$

We denote by $C_y{}^h$ $2m-n$ mutually orthogonal unit normals to M^n, where here and in the sequel, the indices $x, y, z, \cdots$ run over the range $\{n+1, n+2, \cdots, 2m\}$. Then the equations of Gauss are written as

$$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h, \tag{1.4}$$

∇_c being the operator of van der Waerden-Bortolotti covariant differentiation along M^n and $h_{cb}{}^x$ are second fundamental tensors of M^n with respect to the normals $C_x{}^h$, and those of Weingarten as

$$\nabla_c C_x{}^h = -h_c{}^a{}_x B_a{}^h, \tag{1.5}$$

where $h_c{}^a{}_x = h_{cbx} g^{ba} = h_{cb}{}^z g^{ba} g_{zx}$, $(g^{ba}) = (g_{ba})^{-1}$ and g_{zx} denoting the metric tensor of the normal bundle.

If the transform by f of any vector tangent to M^n is always tangent to M^n, that is, if there exists a tensor field $f_b{}^a$ of type $(1, 1)$ such that

$$f_i{}^h B_b{}^i = f_b{}^a B_a{}^h, \tag{1.6}$$

we say that M^n is *invariant* in M^{2m}. This shows that

$$f_{ih} B_b{}^i C_x{}^h = 0.$$

Thus we put

$$f_i{}^h C_y{}^i = f_y{}^x C_x{}^h, \tag{1.7}$$

from which, $f_{yx} = -f_{xy}$, where $f_{yx} = f_y{}^z g_{zx}$. We put

$$\begin{cases} u^h = u^a B_a{}^h + u^x C_x{}^h, \\ v^h = v^a B_a{}^h + v^x C_x{}^h, \end{cases} \tag{1.8}$$

u^a and v^a being vector fields of M^n, u^x and v^x being functions of M^n.

From (1.1), (1.6), (1.7) and (1.8), we find

$$f_b^c f_c^a = -\delta_b^a + u_b u^a + v_b v^a, \tag{1.9}$$

$$f_c^e f_b^d g_{ed} = g_{cb} - u_c u_b - v_c v_b, \tag{1.10}$$

$$f_b^a u^b = -\lambda v^a, \quad f_b^a v^b = \lambda u^a, \tag{1.11}$$

$$u_a u^a = 1 - \lambda^2 - u_x u^x, \quad v_a v^a = 1 - \lambda^2 - v_x v^x, \tag{1.12}$$

$$u_a v^a = -u_x v^x, \tag{1.13}$$

$$u_x u_b = -v_x v_b, \tag{1.14}$$

$$f_x^y f_y^z = -\delta_x^z + u_x u^z + v_x v^z, \tag{1.15}$$

$$u^x f_{xy} = -\lambda v_y, \quad v^x f_{xy} = \lambda u_y. \tag{1.16}$$

We also have from (1.6), $f_{ji} B_c^j B_b^i = f_c^e g_{eb}$.
Thus putting $f_c^e g_{eb} = f_{cb}$, we see that f_{cb} is skew-symmetric. Equations (1.9) $\sim$(1.13) show that a necessary and sufficient condition f_b^a, g_{cb}, u_b, v_b and λ to define an (f, g, u, v, λ)-structure is that

$$u_x = 0, \quad v_x = 0, \tag{1.17}$$

that is, the vector u^h and v^h are always tangent to the submanifold M^n (cf. [4]).

In the sequel we assume that the submanifold M^n has an (f, g, u, v, λ)-structure.

Differentiating (1.6) and (1.7) covariantly along M^n and using (1.2), (1.4) and (1.5), we find

$$\nabla_c f_b^a = g_{cb}(\phi u^a - v^a) - \delta_c^a(\phi u_b - v_b), \tag{1.18}$$

$$\nabla_c f_y^x = 0, \tag{1.19}$$

$$f_b^a h_{ca}^x = h_{cb}^y f_y^x, \tag{1.20}$$

from whch,

$$h_e^e{}_y = 0, \tag{1.21}$$

that is, M^n is minimal and

$$f_b^a h_{ca}^x = f_c^a h_{ba}^x. \tag{1.22}$$

On the other hand, differentiating u^h, v^h and λ covariantly along M^n and using (1.2), (1.6) and (1.4), we get

$$\nabla_c u_a = -\lambda g_{ca} - \phi f_{ca}, \tag{1.23}$$

$$\nabla_c v_a = -\phi\lambda g_{ca} + f_{ca}, \tag{1.24}$$

$$\nabla_c \lambda = u_c + \phi v_c, \tag{1.25}$$

(1.26) $$u^a h_{cax}=0, \quad v^a h_{cax}=0.$$

We put

$$S_{cb}{}^a=[f, f]_{cb}{}^a+(\nabla_c u_b-\nabla_b u_c)u^a+(\nabla_c v_b-\nabla_b v_c)v^a,$$

$[f, f]_{cb}{}^a$ denoting the Nijenhuis tensor formed with $f_b{}^a$. Then we have from (1.18), (1.23) and (1.24)

$$S_{cb}{}^a=0.$$

Thus the (f, g, u, v, λ)-structure induced on M^n is also normal.

Transvecting (1.22) with $f_d{}^b$ and using (1.26), we get

(1.27) $$f_d{}^b f_c{}^a h_{ba}{}^x=-h_{dc}{}^x.$$

Equations of Gauss and Codazzi of the submanifold M^n are respectively given by

(1.28) $$K_{dcb}{}^a=K_{kji}{}^h B_d{}^k B_c{}^j B_b{}^i B^a{}_h+h_d{}^a{}_x h_{cb}{}^x-h_c{}^a{}_x h_{db}{}^x,$$

(1.29) $$K_{kji}{}^h B_d{}^k B_c{}^j B_b{}^i C^x{}_h-(\nabla_d h_{cb}{}^x-\nabla_c h_{db}{}^x)=0,$$

where $K_{dcb}{}^a$ is the curvature tensor of M^n.

Finally, we get from the Ricci-identity, (1.18), (1.23) and (1.24)

(1.30) $$-K_{dcb}{}^e f_e{}^a+K_{dce}{}^a f_b{}^e=(1+\phi^2)\{g_{db}f_c{}^a-f_{cb}\delta_d{}^a-g_{cb}f_d{}^a+\delta_c{}^a f_{db}\}.$$

Transvecting (1.30) with $f_a{}^d$ and using (1.9), (1.12) and (1.17), we have

(1.31) $$K_{dcea}f_b{}^e f^{ad}=-K_{cb}+K_{dcbe}(u^e u^d+v^e v^d) \\ +(1+\phi^2)\{(n-4+2\lambda^2)g_{cb}+2(u_c u_b+v_c v_b)\}.$$

Using the Ricci-identity, (1.23), (1.24), (1.25), (1.18), (1.12) and (1.17), we obtain

(1.32) $$K_{dcbe}(u^e u^d+v^e v^d)=(1+\phi^2)\{2(1-\lambda^2)g_{cb}-(u_c u_b+v_c v_b)\}.$$

Thus we can get the following useful identity from (1.31) and (1.32) for later use

(1.33) $$K_{dcea}f_b{}^e f^{ad}=-K_{cb}+(1+\phi^2)\{(n-2)g_{cb}+u_c u_b+v_c v_b\}.$$

§2. Infinitesimal variations of invariant submanifolds with induced (f, g, u, v, λ)-structure.

We consider an infinitesimal vriation of invariant submanifold M^n with induced (f, g, u, v, λ)-structure of the ambient manifold M^{2m} with normal

(f, g, u, v, λ)-structure given by

$$\bar{x}^h = x^h(y) + \xi^h(y)\varepsilon, \tag{2.1}$$

where $\xi^h(y)$ is a vector field of M^{2m} defined along M^n and ε is an infinitesimal. We then have

$$\bar{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon, \tag{2.2}$$

where $\bar{B}_b{}^h = \partial_b \bar{x}^h$ are n linearly independent vectors tangent to the varied submanifold. We displace $\bar{B}_b{}^h$ parallelly from the varied point $(\bar{x}^h)$ to the original point (x^h). We then obtain the vectors

$$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j \bar{B}_b{}^i \varepsilon \tag{2.3}$$

at the point (x^h), or

$$\tilde{B}_b{}^h = B_b{}^h + (\nabla_b \xi^h)\varepsilon, \tag{2.4}$$

neglecting the terms of order higher than one with respect to ε, where

$$\nabla_b \xi^h = \partial_b \xi^h + \Gamma_{ji}{}^h B_b{}^j \xi^i. \tag{2.5}$$

In the sequel we always neglect terms of order higher than one with respect to ε. Thus putting

$$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h, \tag{2.6}$$

we have from (2.4)

$$\delta B_b{}^h = (\nabla_b \xi^h)\varepsilon. \tag{2.7}$$

If we put

$$\xi^h = \xi^a B_a{}^h + \xi^x C_x{}^h, \tag{2.8}$$

then we get

$$\nabla_b \xi^h = (\nabla_b \xi^a - h_b{}^a{}_x \xi^x) B_a{}^h + (\nabla_b \xi^x + h_{ba}{}^x \xi^a) C_x{}^h \tag{2.9}$$

because of (1.4) and (1.5).

Now we denote by $\bar{C}_y{}^h$ $2m-n$ mutually orthogonal unit normals to the varied submanifold and by $\tilde{C}_y{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by parallel displacement of $\bar{C}_y{}^h$ from the point $(\bar{x}^h)$ to (x^h). Then we have

$$\tilde{C}_y{}^h = \bar{C}_y{}^h + \Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j \bar{C}_y{}^i \varepsilon. \tag{2.10}$$

We put

$$\delta C_y{}^h = \tilde{C}_y{}^h - C_y{}^h \tag{2.11}$$

and assume that $\delta C_y{}^h$ is of the form

$$\delta C_y{}^h = \eta_y{}^h \varepsilon = (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon. \tag{2.12}$$

Then, from (2.10), (2.11) and (2.12), we have

(2.13) $$\bar{C}_y{}^h = C_y{}^h - \Gamma_{ji}{}^h \xi^j C_y{}^i \varepsilon + (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon.$$

Applying the operator δ to $B_b{}^j C_y{}^i g_{ji} = 0$ and using (2.7), (2.9), (2.12) and $\delta g_{ji} = 0$, we find

$$(\nabla_b \xi_y + h_{bay}\xi^a) + \eta_{yb} = 0,$$

where $\xi_y = \xi^z g_{zy}$ and $\eta_{yb} = \eta_y{}^c g_{cb}$, or

(2.14) $$\eta_y{}^a = -(\nabla^a \xi_y + h_b{}^a{}_y \xi^b),$$

∇^a being defined to be $\nabla^a = g^{ac}\nabla_c$. Applying also the operator δ to $C_y{}^j C_x{}^i g_{ji} = g_{yx}$ and using (2.12) and $\delta g_{ji} = 0$, we find

(2.15) $$\eta_{yx} + \eta_{xy} = 0,$$

where $\eta_{yx} = \eta_y{}^z g_{zx}$.

We assume that the infinitesimal variation (2.1) carries an invariant submanifold into an invariant submanifold, that is,

(2.16) $f_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i$ are linear combination of $\bar{B}_b{}^h$.

Now using the first equation of (1.2) and (1.6), we see that

$$\begin{aligned}
& f_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i \\
&= (f_i{}^h + \xi^j \partial_j f_i{}^h \varepsilon)(B_b{}^i + \partial_b \xi^i \varepsilon) \\
&= [f_i{}^h + \xi^j \{-\Gamma_{jk}{}^h f_i{}^k + \Gamma_{ji}{}^k f_k{}^h + g_{ji}(\phi u^h - v^h) \\
&\qquad - \delta_j^h(\phi u_i - v_i)\}\varepsilon][B_b{}^i + (\partial_b \xi^i)\varepsilon] \\
&= f_b{}^a B_a{}^h + [f_i{}^h(\nabla_b \xi^i) - \Gamma_{jk}{}^h \xi^j f_b{}^a B_a{}^k \\
&\qquad + \xi_b(\phi u^h - v^h) - \xi^h(\phi u_b - v_b)]\varepsilon,
\end{aligned}$$

which and (2.2) imply

(2.17) $$\begin{aligned}
& f_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i \\
&= f_b{}^a \bar{B}_a{}^h + [f_i{}^h(\nabla_b \xi^i) - f_b{}^a(\nabla_a \xi^h) \\
&\qquad + \xi_b(\phi u^h - v^h) - \xi^h(\phi u_b - v_b)]\varepsilon,
\end{aligned}$$

or, using (1.6), (1.7), (1.8), (1.17), (1.20), (1.22), (2.8) and (2.9),

(2.18) $$\begin{aligned}
& f_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i = f_b{}^a \bar{B}_a{}^h \\
&\quad + [(\nabla_b \xi^e) f_e{}^a - f_b{}^e(\nabla_e \xi^a) + 2 f_b{}^e h_e{}^a{}_x \xi^x \\
&\quad + \xi_b(\phi u^a - v^a) - \xi^a(\phi u_b - v_b)]\, B_a{}^h \varepsilon \\
&\quad + [f_y{}^x \nabla_b \xi^y - f_b{}^e \nabla_e \xi^x - \xi^x(\phi u_b - v_b)] C_x{}^h \varepsilon.
\end{aligned}$$

Thus (2.16) is equivalent to

$$f_y{}^x(\nabla_b\xi^y)-f_b{}^e(\nabla_e\xi^x)-\xi^x(\phi u_b-v_b)=0. \tag{2.19}$$

An infinitesimal variation given by (2.1) is called an *invariance-preserving* variation if it carries an invariant submanifold into an invariant submanifold. Thus we have

THEOREM 2.1. *In order for an infinitesimal variation to be invariance-preserving, it is necessary and sufficient that the variation vector* ξ^h *satisfies* (2.19).

COROLLARY 2.2. *If a vector field* ξ^h *defines an invariance-preserving variation then another vector field* ξ'^h *which has the same normal part as* ξ^h *has the same property.*

Suppose that an infinitesimal variation given by (2.1) carries a submanifold $x^h=x^h(y)$ into another submanifold $\bar{x}^h=\bar{x}^h(y)$ and the tangent space of the original submanifold at (x^h) and that of the varied submanifold at the corresponding point $(\bar{x}^h)$ are parallel. Then we say that the variation is *parallel* (cf. [7]).

Since we have from (2.6), (2.7) and (2.9)

$$\tilde{B}_b{}^h=[\delta_b{}^a+(\nabla_b\xi^a-h_b{}^a{}_x\xi^x)\varepsilon]B_a{}^h + (\nabla_b\xi^x+h_{ba}{}^x\xi^a)\ C_x{}^h\varepsilon, \tag{2.20}$$

we have the following assertion (cf. [7]):

In order for an infinitesimal variation to be parallel, it is necessary and sufficient that

$$\nabla_b\xi^x+h_{ba}{}^x\xi^a=0. \tag{2.21}$$

§3. The variations of $f_b{}^a$ and $f_y{}^x$.

Suppose that an infinitesimal variation $\bar{x}^h=x^h+\xi^h\varepsilon$ is an invariance-preserving variation. Then putting

$$f_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i=(f_b{}^a+\delta f_b{}^a)\bar{B}_a{}^h, \tag{3.1}$$

we have from (2.18) and (2.19)

$$\delta f_b{}^a=[f_e{}^a\nabla_b\xi^e-f_b{}^e\nabla_e\xi^a+2f_b{}^eh_e{}^a{}_x\xi^x + \xi_b(\phi u^a-v^a)-\xi^a(\phi u_b-v_b)]\varepsilon. \tag{3.2}$$

If an invariance-preserving variation preserves $f_b{}^a$, then we say that it is *f-preserving*.

PROPOSITION 3.1. *An invariance-preserving variation is f-preserving if and only if*

$$(3.3)\qquad (\nabla_b \xi^e) f_e^{\ a} - f_b^{\ e} (\nabla_e \xi^a) + 2 f_b^{\ e} h_e^{\ a}{}_x \xi^x + \xi_b (\phi u^a - v^a) - \xi^a (\phi u_b - v_b) = 0.$$

Now applying the operator δ to (1.3) and using (2.6) and (2.8) and $\delta g_{ji} = 0$, we find (cf. [7])

$$(3.4)\qquad \delta g_{cb} = (\nabla_c \xi_b + \nabla_b \xi_c - 2 h_{cbx} \xi^x) \varepsilon,$$

from which,

$$(3.5)\qquad \delta g^{ba} = -(\nabla^b \xi^a + \nabla^a \xi^b - 2 h^{ba}{}_x \xi^x) \varepsilon.$$

A variation of a submanifold for which $\delta g_{cb} = 0$ is said to be ***isometric.*** We now put

$$(3.6)\qquad \bar{\Gamma}_{cb}^{\ a} = (\partial_c \bar{B}_b^{\ h} + \Gamma_{ji}^{\ h}(\bar{x}) \bar{B}_c^{\ j} \bar{B}_b^{\ i}) \bar{B}^a{}_h$$

and

$$\delta \Gamma_{cb}^{\ a} = \bar{\Gamma}_{cb}^{\ a} - \Gamma_{cb}^{\ a},$$

where $\bar{\Gamma}_{cb}^{\ a}$ are Christoffel symbols of the deformed submanifold.

Substituing (2.2) and (2.20) into (3.6), we then obtain by a straightforward computation,

$$(3.7)\qquad \delta \Gamma_{cb}^{\ a} = [(\nabla_c \nabla_b \xi^h + K_{kji}{}^h \xi^k B_c^{\ i} B_b^{\ i}) B^a{}_h + h_{cb}{}^x (\nabla^a \xi_x + h_d{}^a{}_x \xi^d)] \varepsilon,$$

from which, using equations (1.28) of Gauss and those (1.29) of Codazzi of the submanifolds (cf. [7]), we have

$$(3.8)\qquad \delta \Gamma_{cb}^{\ a} = (\nabla_c \nabla_b \xi^a + K_{dcb}{}^a \xi^d) \varepsilon - [\nabla_c (h_{bex} \xi^x) + \nabla_b (h_{cex} \xi^x) - \nabla_e (h_{cbx} \xi^x)] g^{ea} \varepsilon$$

because of (2.9).

A variation of a submanifold for which $\delta \Gamma_{cb}^{\ a} = 0$ is said to be ***affine.***

Assume that an infinitesimal variation $\bar{x}^h = x^h + \xi^h \varepsilon$ is invarnce-preserving. Hence we have

$$(3.9)\qquad \bar{f}_i^{\ h} \bar{C}_y^{\ i} = \bar{f}_y^{\ x} \bar{C}_x^{\ h}.$$

Then using (2.13), we find

$$(3.10)\qquad f_i^{\ h}(x + \xi\varepsilon)[C_y^{\ i} - \Gamma_{jt}^{\ i} \xi^j C_y^{\ t} \varepsilon + (\eta_y^{\ a} B_a^{\ i} + \eta_y^{\ x} C_x^{\ i}) \varepsilon] = (f_y^{\ x} + \delta f_y^{\ x}) \bar{C}_x^{\ h},$$

or, using (1.2), (1.6), (1.7) and (1.17),

$$f_y{}^x\overline{C}_x{}^h - f_y{}^x(\eta_x{}^a B_a{}^h + \eta_x{}^z C_z{}^h)\varepsilon$$
$$+ (\eta_y{}^b f_b{}^a B_a{}^h + \eta_y{}^x f_x{}^z C_z{}^h)\varepsilon + \xi_y(\phi u^a - v^a) B_a{}^h\varepsilon$$
$$= f_y{}^x\overline{C}_x{}^h + (\delta f_y{}^x)\overline{C}_x{}^h,$$

from which, we have

$$\delta f_y{}^x = (-f_y{}^w\eta_w{}^x + \eta_y{}^w f_w{}^x)\varepsilon, \tag{3.11}$$

$$f_y{}^x\eta_x{}^a = \eta_y{}^b f_b{}^a + \xi_y(\phi u^a - v^a), \tag{3.12}$$

or, using (2.14) and (1.20),

$$f_y{}^x(\nabla^a\xi_x) = (\nabla^b\xi_y) f_b{}^a - \xi_y(\phi u^a - v^a). \tag{3.13}$$

PROPOSITION 3.2. *Suppose that an infinitesimal variation is invariance-preserving. Then the variation of* $f_y{}^x$ *is given by* (3.11).

§4. Variations of u^a, v^a and λ.

In this section we compute the variations of u^a, v^a and λ on the submanifold.

Now we get a vector field $\bar{u}^h$ which is defined intrinsically along the deformed submanifold. If we displace $\bar{u}^h$ back parallelly from the point $(\bar{x})^h$ to $(x)^h$, we obtain

$$\tilde{u}^h = \bar{u}^h + \Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j\bar{u}^i$$

and hence forming

$$\delta u^h = \tilde{u}^h - u^h, \tag{4.1}$$

we find

$$\delta u^h = \bar{u}^h - u^h + \Gamma_{ji}{}^h\xi^i u^i\varepsilon. \tag{4.2}$$

We have, from (1.8) and (4.2),

$$\delta(u^a B_a{}^h + u^x C_x{}^h) = \xi^k(\partial_k u^h)\varepsilon + \Gamma_{ji}{}^h\xi^j u^i\varepsilon, \tag{4.3}$$

which and (1.2), (2.7) and (2.12) imply

$$(\delta u^a) B_a{}^h + (\delta u^x) C_x{}^h$$
$$= -[u_b(\nabla^b\xi^a - h^{ba}{}_x\xi^x) + u^y\eta_y{}^a + \lambda\xi^a + \phi\xi^b f_b{}^a] B_a{}^h\varepsilon$$
$$-[u_b(\nabla^b\xi^x + h^b{}_a{}^x\xi^a) + u^y\eta_y{}^x + \lambda\xi^x + \phi\xi^y f_y{}^x] C_x{}^h\varepsilon.$$

Then using (1.17) and (1.26), we get

$$\delta u^a = -(\phi\xi^b f_b{}^a + \lambda\xi^a + u^b \nabla_b \xi^a)\varepsilon, \tag{4.4}$$

from which, using (3.4),

$$\delta u_a = -(\phi\xi^b f_{ba} + \lambda\xi_a - u^b \nabla_a \xi_b)\varepsilon. \tag{4.5}$$

Thus we have

PROPOSITION 4.1. *Under an infinitesimal variation* (2.1) *of the submanifold, the variation of* u^a, u_a *are given by* (4.4) *and* (4.5) *respectively.*

Similarly we get a vector field $\bar{v}^h$ which is defined intrinsically along the deformed submanifold. If we displace $\bar{v}^h$ back parallelly from the point $(\bar{x}^h)$ to (x^h), we obtain

$$\tilde{v}^h = \bar{v}^h + \Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j \bar{v}^i \varepsilon$$

and hence forming

$$\delta v^h = \tilde{v}^h - v^h, \tag{4.6}$$

we find

$$\delta v^h = \bar{v}^h - v^h + \Gamma_{ji}{}^h \xi^j v^i \varepsilon. \tag{4.7}$$

We have from (1.8) and (4.7)

$$\delta(v^a B_a{}^h + v^x C_x{}^h) = \xi^k(\partial_k v^h)\varepsilon + \Gamma_{ji}{}^h \xi^j v^i \varepsilon, \tag{4.8}$$

which and (1.2), (2.7) and (2.12) imply

$$\begin{aligned}(\delta v^a)B_a{}^h + (\delta v^x)C_x{}^h \\ &= [-v^b(\nabla_b\xi^a - h_b{}^a{}_x\xi^x) - v^y\eta_y{}^a - \phi\lambda\xi^a + \xi^b f_b{}^a]B_a{}^h\varepsilon \\ &+ [-v^b(\nabla_b\xi^x + h_{ba}{}^x\xi^a) - v^y\eta_y{}^x - \phi\lambda\xi^x + \xi^y f_y{}^x]C_x{}^h\varepsilon.\end{aligned}$$

Then using (1.17) and (1.26), we obtain

$$\delta v^a = (\xi^b f_b{}^a - \phi\lambda\xi^a - v^b\nabla_b\xi^a)\varepsilon, \tag{4.9}$$

from which, using (3.4),

$$\delta v_a = (\xi^b f_{ba} - \phi\lambda\xi_a + v^b\nabla_a\xi_b)\varepsilon. \tag{4.10}$$

Thus we have

PROPOSITION 4.2. *Under an infinitesimal variation* (2.1) *of the submanifold, the variations of* v^a *and* v_a *are given by* (4.9) *and* (4.10) *respectively.*

Finally, to obtain the variation of λ, applying the operator δ to $u^a u_a = 1-\lambda^2$ and using (4.4) and (4.5), we have

$$\delta\lambda = (u_a + \phi v_a)\xi^a\varepsilon. \tag{4.11}$$

Thus we have

PROPOSITION 4.3. *Under an infinitesimal variation* (2.1) *of the submanifold, the variation of* λ *is given by* (4.11).

Furthermore we have from (1.9)

$$\delta(f_b{}^a f_a{}^c) = (\delta u_b)u^c + u_b(\delta u^c) + (\delta v_b)v^c + v_b(\delta v^c). \tag{4.12}$$

If the variation preserves $f_b{}^a$ and u^a, we have from (4.12)

$$(\delta u_b)u^c + (\delta v_b)v^c + v_b(\delta v^c) = 0. \tag{4.13}$$

Transvecting (4.13) with u_c and v_c, we find respectively

$$(1-\lambda^2)\delta u_b + u_c v_b(\delta v^c) = 0, \tag{4.14}$$

$$(1-\lambda^2)\delta v_b + v_c v_b(\delta v^c) = 0. \tag{4.15}$$

Transvecting (4.14) with u^b, $(\delta u_b)u^b=0$. Then

$$\delta(u^b u_b) = -2\lambda(\delta\lambda) = 0, \quad \text{that is,} \quad \delta\lambda = 0.$$

Applying the operator δ to (1.11), we can get from above $\delta v^a=0$. So from (4.14) and (4.15), $\delta u_b=0$ and $\delta v_b=0$. Thus we have

PROPOSITION 4.4. *If an infinitesimal f-preserving variation preserves* u^a, *then the variation preserves* u_a, v^a, v_a *and* λ.

§5. An integral formula.

In this section, we calculate $T^{cb}T_{cb}$, T_{cb} being defined in such a way that the variation is f-preserving if and only if $T_{cb}=0$. In the later part of this section, we consider an infinitesimal invariance-preserving variation of a compact invariant submanifold with induced (f, g, u, v, λ)-structure of the even dimensional sphere S^{2m} with normal (f, g, u, v, λ)-structure and get an integral formula involving $T_{cb}T^{cb}$.

First of all, we define T_{cb} by

$$\begin{aligned} T_{cb} = {} & (\nabla_c \xi^e) f_{eb} - f_c{}^e(\nabla_e \xi_b) + 2 f_c{}^e h_{ebx}\xi^x \\ & + \xi_c(\phi u_b - v_b) - \xi_b(\phi u_c - v_c). \end{aligned} \tag{5.1}$$

Then we find that a variation of invariant submanifold is f-preserving if and only if $T_{cb}=0$.

If we take account of (1.9), (1.11), (1.12), (1.17), (1.26) and (1.27), we have

$$T_{cb}T^{cb} = 2(\nabla_c \xi_a)(\nabla^c \xi^a) - 8h_{cbx}\xi^x(\nabla^c \xi^b) \tag{5.2}$$

$$+4(h_{cbx}\xi^x)(h^{cb}{}_y\xi^y)$$
$$-u^e u_a[(\nabla_c\xi_e)(\nabla^c\xi^a)+(\nabla_e\xi_c)(\nabla^a\xi^c)]$$
$$-v^e v_a[(\nabla_c\xi_e)(\nabla^c\xi^a)+(\nabla_e\xi_c)(\nabla^a\xi^c)]$$
$$+2\phi\lambda\xi_c v_a(\nabla^c\xi^a-\nabla^a\xi^c)+2\lambda\xi_c u_a(\nabla^c\xi^a-\nabla^a\xi^c)$$
$$-2f_c{}^e f_a{}^b(\nabla_e\xi_b)(\nabla^c\xi^a)$$
$$-2f_a{}^b\xi_b(\phi u_c-v_c)(\nabla^c\xi^a-\nabla^a\xi^c)$$
$$+2(1-\lambda^2)(1+\phi^2)\xi_c\xi^c-2\phi^2(\xi_c u^c)^2$$
$$+2\phi(\xi_c u^c)(\xi_a v^a)-2(\xi_c v^c)^2.$$

On the other hand, we have

$$\nabla_b w^b=(\nabla_b\nabla^b\xi^c)\xi_c+(\nabla^b\xi^c)(\nabla_b\xi_c)$$
$$-(n-1)(\phi u^e-v^e)f^{ac}\xi_a(\nabla_e\xi_c)$$
$$-f^{be}\xi_b(\phi u^c-v^c)(\nabla_e\xi_c)+f^{be}(\phi u^a-v^a)(\nabla_e\xi_b)\xi_a$$
$$-f^{be}f^{ac}(\nabla_b\nabla_e\xi_c)\xi_a-f^{be}f^{ac}(\nabla_e\xi_c)(\nabla_b\xi_a) \tag{5.3}$$

because of (1.18), where we have put

$$w^b=(\nabla^b\xi^c)\xi_c-f^{be}f^{ac}(\nabla_e\xi_c)\xi_a,$$

from which, using the Ricci identity and (1.33),

$$\nabla_b W^b=(\nabla_b\nabla^b\xi^c)\xi_c+(\nabla^b\xi^c)(\nabla_b\xi_c)$$
$$-(n-1)(\phi u^e-v^e)(\nabla_e\xi_c)f^{ac}\xi_a$$
$$-f^{be}\xi_b(\phi u^c-v^c)(\nabla_e\xi_c)+f^{be}(\phi u^a-v^a)\xi_a(\nabla_e\xi_b)$$
$$+K_{da}\xi^d\xi^a-(1+\phi^2)(n-2)\xi_a\xi^a$$
$$-(1+\phi^2)(u_a\xi^a)^2$$
$$-(1+\phi^2)(v_a\xi^a)^2-f^{be}f^{ac}(\nabla_e\xi_c)(\nabla_b\xi_a). \tag{5.4}$$

Comparing (5.2) with (5.4), we have

$$T^{cb}T_{cb}=2\nabla_b W^b-2\xi^c(\nabla^b\nabla_b\xi_c+K_{cb}\xi^b)$$
$$-8(h_{cbx}\xi^x)(\nabla^c\xi^b)+4(h_{cbx}\xi^x)(h^{cb}{}_y\xi^y)$$
$$-(u^e u_a+v^e v_a)\ [(\nabla_c\xi_e)(\nabla^c\xi^a)+(\nabla_e\xi_c)(\nabla^a\xi^c)]$$
$$+2\phi\lambda\xi_c v_a(\nabla^c\xi^a-\nabla^a\xi^c)+2\lambda\xi_c u_a(\nabla^c\xi^a-\nabla^a\xi^c)$$
$$+2(1+\phi^2)(n-1-\lambda^2)\xi_a\xi^a+2(u_a\xi^a)^2+2\phi^2(v^a\xi_a)^2 \tag{5.5}$$

$$+2\phi(\xi^c u_c)(\xi^a v_a)-2f^{be}(\phi u^a-v^a)\xi_a(\nabla_e\xi_b)$$
$$+2nf^{ac}\xi_a(\phi u^e-v^e)(\nabla_e\xi_c),$$

or, equivalently

$$\begin{aligned}(5.6)\qquad T^{cb}T_{cb}=&2\nabla^b(W_b-2h_{cbx}\xi^c\xi^x)\\&-2\xi^c[\nabla^b\nabla_b\xi_c+K_{cb}\xi^b-2\nabla^b(h_{cbx}\xi^x)]\\&-2(h^{cb}{}_y\xi^y)(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)\\&-(u^e u_a+v^e v_a)[(\nabla_c\xi_e)(\nabla^c\xi^a)+(\nabla_e\xi_c)(\nabla^a\xi^c)]\\&+2\phi\lambda\xi_c v_a(\nabla^c\xi^a-\nabla^a\xi^c)+2\lambda\xi_c u_a(\nabla^c\xi^a-\nabla^a\xi^c)\\&+2(1+\phi^2)(n-1-\lambda^2)\xi_a\xi^a+2(u_a\xi^a)^2\\&+2\phi^2(v_a\xi^a)^2+2\phi(\xi^c u_c)(\xi^a v_a)\\&-2f^{be}(\phi u^a-v^a)\xi_a(\nabla_e\xi_b)+2nf^{be}\xi_b(\phi u^c-v^c)(\nabla_c\xi_e).\end{aligned}$$

Now we assume that the ambient manifold is an even dimensional sphere S^{2m} and that M^n is a compact invariant submanifold with induced (f, g, u, v, λ)-structure of S^{2m}. Moreover an even dimensional sphere S^{2m} induces a normal (f, g, u, v, λ)-structure and satisfies differential equations of theorem 1.1 with $\phi=0$ (cf. [1]).

Thus using $\nabla_e(f^{be}v^a\xi_a\xi_b)=-\xi^a\xi_a+(u_a\xi^a)^2+n(v_a\xi^a)^2+f^{be}\xi_b v^a(\nabla_e\xi_a)$
$$+f^{be}v^a\xi_a(\nabla_e\xi_b)$$

and (5.6), we apply Green's theorem and obtain

$$\begin{aligned}(5.7)\qquad \int[&T^{cb}T_{cb}+2\xi^c\{\nabla^b\nabla_b\xi_c+K_{cb}\xi^b-2\nabla^b(h_{cbx}\xi^x)\}\\&+2h^{cb}{}_y\xi^y(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)\\&+(u^e u_a+v^e v_a)\{(\nabla_c\xi_e)(\nabla^c\xi^a)+(\nabla_e\xi_c)(\nabla^a\xi^c)\}\\&-2\lambda\xi_c u_a(\nabla^c\xi^a-\nabla^a\xi^c)-2(n-\lambda^2)\xi^a\xi_a\\&+2n(v_a\xi^a)^2+2f^{be}\xi_b v^c(\nabla_e\xi_c)\\&+2nf^{be}\xi_b v^c(\nabla_c\xi_e)]dV=0,\end{aligned}$$

dV being the volume element of M^n.

From (3.4) and (3.5), the variation of dV is given by (cf. [7])

$$(5.8)\qquad \delta dV=(\nabla_a\xi^a-h_a{}^a{}_x\xi^x)dV\varepsilon.$$

For a compact orientable submanifold M^n, we have the following integral formula:

$$\int[\xi^c(\nabla^b\nabla_b\xi_c+K_{cb}\xi^b)$$
$$+\frac{1}{2}(\nabla_c\xi_b+\nabla_b\xi_c)(\nabla^c\xi^b+\nabla^b\xi^c)-(\nabla_b\xi^b)^2]dV=0,$$

which is valid for any vector ξ^c in M^n (cf. [6], [9]), from which

(5.9)
$$\int[\xi^c\{(\nabla^b\nabla_b\xi_c+K_{cb}\xi^b)-2\nabla^b(h_{cb}{}^x\xi_x)+\nabla_c(h_b{}^b{}_x\xi^x)\}$$
$$+\frac{1}{2}(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cby}\xi^y)(\nabla^c\xi^b+\nabla^b\xi^c-2h^{cb}{}_x\xi^x)$$
$$-(\nabla_c\xi^c-h_c{}^c{}_x\xi^x)(\nabla_b\xi^b)$$
$$+(h^{cb}{}_y\xi^y)(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)]dV=0.$$

Thus we have an integral formula from (1.21), (5.7) and (5.9)

(5.10)
$$\int[-T^{cb}T_{cb}+(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cby}\xi^y)(\nabla^c\xi^b+\nabla^b\xi^c-2h^{cb}{}_x\xi^x)$$
$$-2(\nabla_c\xi^c)^2+2\lambda\xi_c u_a(\nabla^c\xi^a-\nabla^a\xi^c)$$
$$-(u^e u_a+v^e v_a)\{(\nabla_c\xi_e)(\nabla^c\xi^a)+(\nabla_e\xi_c)(\nabla^a\xi^c)\}$$
$$-2nf^{be}\xi_b v^c(\nabla_c\xi_e)-2f^{be}\xi_b v^c(\nabla_e\xi_c)$$
$$+2(n-\lambda^2)\xi_a\xi^a-2n(v_a\xi^a)^2]dV=0.$$

§6. Isometric and f-preserving variations.

Similar to the later part of §5, we consider in this section an infinitesimal invariance-preserving variation of a compact invariant submanifold with induced (f, g, u, v, λ)-structure of the ambient manifold S^{2m} with normal (f, g, u, v, λ)-structure. Using some integral formulas involving $T_{cb}T^{cb}$, we prove theorems on f-preserving and isometric variations.

We assume that an invariance-preserving variation of the submanifolds preserves u^a, v^a, u_a and v_a. Then we have from (4.4), (4.5) (4.9) and (4.10)

(6.1) $$\lambda\xi^a=-u^b\nabla_b\xi^a,\quad \lambda\xi_a=u^b\nabla_a\xi_b,$$

(6.2) $$\xi_b f^{ba}=v^b\nabla_b\xi^a,\quad \xi^b f_{ba}=-v^b\nabla_a\xi_b,$$

from which, we get

(6.3) $$u^b\nabla_a\xi_b=-u^b\nabla_b\xi_a,\quad v^b\nabla_a\xi_b=-v^b\nabla_b\xi_a.$$

Transvecting the later equations of (6.1) and (6.2) with ξ^a, we find

(6.4) $$\xi^a u^b(\nabla_a\xi_b)=\lambda\xi_a\xi^a,\quad \xi^a v^b(\nabla_a\xi_b)=0.$$

From (4.11), we obtain

(6.5) $$u^a\xi_a=0.$$

Using equations (5.10), (6.2), (6.3) and (6.4), we get

(6.6) $$\begin{aligned}\int[&-T^{cb}T_{cb}+(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cb}{}^y\xi_y)(\nabla^c\xi^b+\nabla^b\xi^c-2h^{cb}{}_x\xi^x)\\&-2(\nabla_c\xi^c)^2-2u^eu_a(\nabla_c\xi_e)(\nabla^c\xi^a)\\&-2nv^cv_b(\nabla^b\xi^e)(\nabla_c\xi_e)\\&+2(n+\lambda^2)\xi^a\xi_a-2n(v^a\xi_a)^2]dV=0.\end{aligned}$$

On the other hand we have from (1.9), (6.2) and (6.5)

(6.7) $$v^av_c(\nabla_a\xi_b)(\nabla^c\xi^b)=\xi^a\xi_a-(\xi^av_a)^2,$$

and from (6.1),

(6.8) $$u^au_b(\nabla_c\xi_a)(\nabla^c\xi^b)=\lambda^2\xi^a\xi_a.$$

Thus we get from equations (6.6), (6.7) and (6.8)

(6.9) $$\begin{aligned}\int[&-T^{cb}T_{cb}+(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cby}\xi^y)(\nabla^c\xi^b+\nabla^b\xi^c-2h^{cb}{}_x\xi^x)\\&-2(\nabla_c\xi^c)^2]dV=0.\end{aligned}$$

From this integral formula, (1.21) and (5.8), we have

PROPOSITION 6.1. *Suppose that the ambient manifold is an even dimensional sphere with normal (f, g, u, v, λ)-structure and an invariance-preserving variation preserves u^a, v^a, u_a and v_a.*

Then in order for an invariance-preserving variation of a compact invariant submanifold with induced (f, g, u, v, λ)-structure to be isometric it is necessary and sufficient that the variation is volume-preserving and f-preserving.

Furthermore, if a variation of the submanifold is affine, we have from (3.8)

$$\begin{aligned}\nabla_c\nabla_b\xi_a+K_{dcba}\xi^d-\nabla_c(h_{bax}\xi^x)-\nabla_b(h_{cax}\xi^x)\\+\nabla_a(h_{cbx}\xi^x)=0,\end{aligned}$$

from which, using (1.21)

$$\nabla_c(\nabla_a\xi^a)=0,$$

that is, $\nabla_a\xi^a=$constant. Thus assuming the submanifold to be compact, we have $\nabla_a \xi^a = 0$. From this fact and proposition 6.1, we obtain

THEOREM 6.2. *Assume that the ambient manifold is an even dimensional sphere with normal (f, g, u, v, λ)-structure and an invariance-preserving variation preserves u^a, v^a, u_a and v_a. Then an invariance-preserving variation of a compact invariant submanifold with induced (f, g, u, v, λ)-structure is isometric if and only if the variation is affine and f-preserving.*

On the other hand, if the isometric variation preserves u^a and v^a, then the variation is affine and $\delta u_a = 0$ and $\delta v_a = 0$. Thus we have

COROLLARY 6.3. *Suppose that the ambient manifold is an even dimensional sphere with normal (f, g, u, v, λ)-structure. If the submanifold with induced (f, g, u, v, λ)-structure is compact, and an invariance-preserving isometric variation preserves u^a and v^a, then the variation is f-preserving.*

Moreover, we have from proposition 4.4

COROLLARY 6.4. *Suppose that the ambient manifold is an even dimensional sphere with normal (f, g, u, v, λ)-structure and an invariance-preserving variation is f-preserving and preserves u^a. Then the variation of a comact invariant submanifold with induced (f, g, u, v, λ)-structure is affine if and only if it is isometric.*

Now we assume that an infinitesimal variation is fibre-preserving u^a, u_a, v^a and v_a, that is,

$$\text{(6.10)} \qquad \begin{cases} \mathcal{L}_\xi u^a = \alpha u^a, & \mathcal{L}_\xi u_a = \beta u_a, \\ \mathcal{L}_\xi v^a = \mu v^a, & \mathcal{L}_\xi v_a = \nu v_a, \end{cases}$$

where α, β, μ and ν are functions on submanifold M^n. Then we have from (6.10)

$$\text{(6.11)} \qquad u^b \nabla_b \xi^a = -\alpha u^a - \lambda \xi^a,$$

$$\text{(6.12)} \qquad u^b \nabla_a \xi_b = \beta u_a + \lambda \xi_a,$$

$$\text{(6.13)} \qquad \xi^b f_b{}^a = v^b \nabla_b \xi^a + \mu v^a,$$

$$\text{(6.14)} \qquad \xi^b f_{ba} = -v^b \nabla_a \xi_b + \nu v_a,$$

from which, we get

$$\text{(6.15)} \qquad \alpha u^a + u^b \nabla_b \xi^a = \beta u^a - u_b \nabla^a \xi^b,$$

$$\text{(6.16)} \qquad \mu v_a + v^b \nabla_b \xi_a = \nu v_a - v^b \nabla_a \xi_b.$$

Transvecting (6.15) with v_a, we find

(6.17) $$u^b v^a (\nabla_b \xi_a) = -u^b v^a (\nabla_a \xi_b).$$

Transvecting (6.11), (6.13) with ξ_a and (6.12), (6.14) with ξ^a, we get

(6.18) $$\xi^a u^b (\nabla_b \xi_a) = -\lambda \xi^a \xi_a - \alpha u^a \xi_a,$$

$$\xi^a u^b (\nabla_a \xi_b) = \lambda \xi^a \xi_a + \beta u^a \xi_a,$$

(6.19) $$\xi^a v^b (\nabla_b \xi_a) = -\mu v^a \xi_a, \quad \xi^a v^b (\nabla_a \xi_b) = \nu v^a \xi_a.$$

Thus we have from (5.10), (6.13) and (6.18)

(6.20) $$\begin{aligned}\int[&-T^{cb}T_{cb} + (\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cby}\xi^y)(\nabla^c \xi^b + \nabla^b \xi^c - 2h^{cb}{}_x \xi^x)\\ &-2(\nabla_c \xi^c)^2 + 2(n+\lambda^2)\xi^a \xi_a\\ &-(u^e u_a + v^e v_a)\{(\nabla_c \xi_e)(\nabla^c \xi^a) + (\nabla_e \xi_c)(\nabla^a \xi^c)\}\\ &+2\beta\lambda(u^a \xi_a) + 2\alpha\lambda(u^a \xi_a) - 2n(v^a \xi_a)^2\\ &-2(n+1)\mu v^c v^e (\nabla_c \xi_e) - 2n v^c v^b (\nabla_c \xi_e)(\nabla_b \xi^e)\\ &-2v^c v_b (\nabla_e \xi_c)(\nabla^b \xi^e)]dV = 0.\end{aligned}$$

Transvecting (6.11), (6.12) with u_a, u^a and (6.13), (6.14) with v_a, v^a respectively, we get the following equations:

(6.21) $$u^b u^a (\nabla_b \xi_a) = -\alpha(1-\lambda^2) - \lambda \xi^a u_a,$$

(6.22) $$u^b u^a (\nabla_b \xi_a) = \beta(1-\lambda^2) + \lambda \xi^a u_a,$$

(6.23) $$v^b v^a (\nabla_b \xi_a) = -\mu(1-\lambda^2) - \lambda \xi^a u_a,$$

(6.24) $$v^b v^a (\nabla_b \xi_a) = \nu(1-\lambda^2) + \lambda \xi^a u_a,$$

from which,

(6.25) $$\xi^a u_a = \frac{1}{2\lambda}[-\alpha(1-\lambda^2) - \beta(1-\lambda^2)],$$

(6.26) $$\xi^a u_a = \frac{1}{2\lambda}[-\mu(1-\lambda^2) - \nu(1-\lambda^2)].$$

Moreover from (6.11), (6.12), (6.13) and (6.14), we find

(6.27) $$\begin{cases} u_b u^c (\nabla^b \xi^a)(\nabla_c \xi_a) = \alpha^2(1-\lambda^2) + 2\alpha\lambda \xi^a u_a + \lambda^2 \xi^a \xi_a,\\ u_b u^c (\nabla^a \xi^b)(\nabla_a \xi_c) = \beta^2(1-\lambda^2) + 2\beta\lambda \xi^a u_a + \lambda^2 \xi^a \xi_a,\\ v^c v_b (\nabla_c \xi_a)(\nabla^b \xi^a) = \xi^b \xi_b - (\xi^b u_b)^2 - (\xi^b v_b)^2\\ \qquad + \mu^2(1-\lambda^2) + 2\mu\lambda \xi^a u_a,\end{cases}$$

$$\begin{cases} v^c v_b(\nabla_a \xi_c)(\nabla^b \xi^a) = -\xi^b \xi_b + (\xi^b u_b)^2 + (\xi^b v_b)^2 \\ \qquad -\mu\lambda\xi^c u_c - \nu\lambda\xi^c u_c - \mu\nu(1-\lambda^2), \\ v^c v_b(\nabla_a \xi_c)(\nabla^a \xi^b) = \xi^a \xi_a - (\xi^a u_a)^2 - (\xi^a v_a)^2 \\ \qquad + 2\nu\lambda\xi^a u_a + \nu^2(1-\lambda^2). \end{cases}$$

Using equations (6.20), (6.23) and (6.27), we get

$$(6.28)\quad \int[-T^{cb}T_{cb} + (\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cby}\xi^y)(\nabla^c \xi^b + \nabla^b \xi^c - 2h^{cb}{}_x \xi^x) \\ -2(\nabla_c \xi^c)^2 - \alpha^2(1-\lambda^2) - \beta^2(1-\lambda^2) + \mu^2(1-\lambda^2) - \nu^2(1-\lambda^2) \\ -2(n-1)\mu\lambda\xi^a u_a + 2n(\xi^a u_a)^2 + 2\mu\nu(1-\lambda^2)]dV = 0.$$

On the other hand, we have from (1.9), (6.13), (6.24) and (6.27)

$$\xi^b f_b{}^a \xi^c f_{ca} = \xi^b \xi_b - (\xi^b u_b)^2 - (\xi^b v_b)^2,$$

$$\xi^b f_b{}^a \xi^c f_{ca} = \xi^b \xi_b - (\xi^b u_b)^2 - (\xi^b v_b)^2 \\ + 2\mu^2(1-\lambda^2) + 2\mu\nu(1-\lambda^2) \\ + 4\mu\lambda\xi^a u_a,$$

from which

$$(6.29)\quad \mu^2(1-\lambda^2) + \mu\nu(1-\lambda^2) + 2\mu\lambda\xi^a u_a = 0.$$

Thus we have from (6.28) and (6.29))

$$(6.30)\quad \int[-T_{cb}T^{cb} + (\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cby}\xi^y)(\nabla^c \xi^b + \nabla^b \xi^c - 2h^{cb}{}_x \xi^x) \\ -2(\nabla_c \xi^c)^2 - \alpha^2(1-\lambda^2) - \beta^2(1-\lambda^2) - \mu^2(1-\lambda^2) - \nu^2(1-\lambda^2) \\ -2(n+1)\mu\lambda(\xi^a u_a) + 2n(\xi^a u_a)^2]dV = 0.$$

We assume next that the variation of a compact invariant submanifold is isometric. Then we have from (1.26), (3.4), (6.11)~(6.14), (6.25) and (6.26)

$$(6.31)\quad \alpha = \beta = \mu = \nu.$$

And we get from (1.23) and (1.25)

$$\nabla^c(\xi^a u_a) = u_a \nabla^c \xi^a - \lambda\xi^c,$$

$$\triangle(\xi^a u_a) = -2\lambda\nabla_c \xi^c + u^a \nabla^c \nabla_c \xi_a - u_c \xi^c.$$

Moreover if the variation on a compact submanifold is isometric, we have

$\nabla_c \xi^c \doteq 0$ with the help of (1.21) and (3.8), from which,

(6.32) $$\nabla^c(\xi^a u_a) = u_a \nabla^c \xi^a - \lambda \xi^c,$$

(6.33) $$\Delta(\xi^a u_a) = -u^c \xi_c + u_a \nabla_c \nabla^c \xi^a.$$

Now we find from the equations of Gauss (1.28) and Codazzi (1.29)

(6.34) $$K_{cb} \xi^c u^b = (n-1) \xi^c u_c,$$

(6.35) $$\nabla_d h_{cb}{}^x = \nabla_c h_{db}{}^x.$$

Since isometric variation is affine, we have from (3.8), (1.21) and (6.35)

$$u^a \nabla^c \nabla_c \xi_a + K_{da} \xi^d u^a = 0,$$

from which, using (6.33) and (6.34)

$$\Delta(\xi^a u_a) = -n \xi^c u_c.$$

Hence if $\alpha\lambda$ has definite sign and the submanifold is compact, we have from (6.25), (6.26) and (6.31)

$$\xi^c u_c = 0 \text{ and } \alpha = \beta = \mu = \nu = 0,$$

from which, using (4.4), (4.5), (4.9), (4,10) and (6.11)～(6.14)

$$\delta u^a = \delta u_a = \delta v^a = \delta v_a = 0.$$

From these facts and (6.30), we have

THEOREM 6.5. *Suppose that an invariance-preserving isometric variation of a compact invariant submanifold with induced (f, g, u, v, λ)-structure of the even dimensional sphere with normal (f, g, u, v, λ)-structure is fibre-preserving u^a and v^a such that $\pounds_\xi u^a = \alpha u^a$ and $\pounds_\xi v^a = \mu v^a$, where α and μ are functions on the submanifold. Then if $\alpha\lambda$ has definite sign, the variation is f-preserving and preserves u^a, u_a, v^a and v_a.*

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