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DISTAL HOMOMORPHISMS

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In this paper, we will show that for a bitransformation group (H, X, T), the natural homomorphism from the transformation group (X, T) to the transformation group $(X/H, T)$ is distal. The pointwise almost periodicity can be lifted by distal homomorphisms. By this result, we give a very direct proof of a result of Ellis. Finally, we will show that if an epimorphism between transformation groups is distal, then the epimorphism between the enveloping semigroups is also distal, but the converse does not hold.

In this paper, let T be an arbitrary, but fixed, topological group and we consider (right) transformation groups (X, T) (or left transformation groups (H, X)) with a compact Hausdorff phase space X. A closed nonempty subset A of X is said to be a *minimal set* if, for every $x \in A$, the orbit xT is a dense subset of A. A point whose orbit closure is a minimal set is called an *almost periodic point*. If X is itself minimal, we say it is a *minimal transformation group*. The transformation group (X, T) is called *pointwise almost periodic* if, for each $x \in X$, x is an almost periodic point.

The points x and y of X in the transformation group (X, T) are said to be *proximal* if there exists a net (t_α) of elements of T such that $\lim xt_\alpha = \lim yt_\alpha$. We denote $P(X, T) = \{(x, y) \in X \times X \mid x \text{ and } y \text{ are proximal}\}$ which is called the *proximal relation* on (X, T). The transformation group (X, T) is said to be *distal* if $x, y \in X$ and $(x, y) \in P(X, T)$ imply $x = y$.

If (Y, T) is also a transformation group, a *homomorphism* from (X, T) to (Y, T) is a continuous map $\phi : X \longrightarrow Y$ such that $\phi(xt) = \phi(x)t$ $(x \in X, t \in T)$. In this context the meaning of epimorphisms, isomorphisms and automorphisms is clear.

A homomorphism $\phi : (X, T) \longrightarrow (Y, T)$ is said to be *distal* [1] provided $\phi(x) = \phi(x')$ and $(x, x') \in P(X, T)$ imply $x = x'$.

As is customary, let X^X denote the set of all functions from X to X, provided with the topology of pointwise convergence, and consider T as the subset $\{t : x \longrightarrow xt \ t \in T\}$ of X^X. The *enveloping semigroup* $E(X)$ of the transformation group (X, T) is the closure of T in X^X. Then $E(X)$ is a compact Hausdorff space and we may consider $(E(X), T)$ as a transforma-

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tion group, whose phase space $E(X)$ admits a semigroup structure. The *minimal right ideals* I of $E(X)$ (that is, the nonempty subsets I of $E(X)$ such that $IE(X) \subset I$, which contains no proper nonempty subsets with the same property) coincide with the minimal sets in the transformation group $(E(X), T)$ (see 3.4 [3]).

A *bitransformation group* is a pair of a left transformation group (H, X) and a right transformation group (X, T) with the same phase space X such that $h(xt) = (hx)t$ $(h \in H, x \in X, t \in T)$. The notation (H, X, T) will be used to signify that the pair (H, X), (X, T) constitute a bitransformation group.

In [2], the class of regular minimal transformation groups was introduced. One characterization of a regular minimal transformation group (X, T) is: if $x, x' \in X$, then there is an endomorphism ϕ of (X, T) such that $\phi(x)$ and x' are proximal.

THEOREM 1. *Let (H, X, T) be a bitransformation group such that X/H is Hausdorff and $(X/H, T)$ be pointwise almost periodic. Then the natural map $\phi: (X, T) \longrightarrow (X/H, T)$ is a distal homomorphism.*

Proof. Let $\phi(x) = \phi(y)$. Since $(X/H, T)$ is pointwise almost periodic, $\phi(x)$ is an almost periodic point of $(X/H, T)$. By Proposition 6.1 in [3], there exists an almost periodic point z of (X, T) such that $\phi(z) = \phi(x)$. Thus x and y belong to the same class of z (that is, $\tilde{x} = \tilde{y} = \tilde{z}$). Hence we let $x = h'z$ and $y = hz$.

Suppose $(x, y) \in P(X, T)$, then there exists a net (t_α) such that $\lim\,(h'z)t_\alpha = \lim\,(hz)t_\alpha$.

Since h and h' are automorphisms from (X, T) to (X, T) and we may assume $\lim zt_\alpha$ exists, we have

$$h(\lim zt_\alpha) = h'(\lim zt_\alpha).$$

If we let $w = \lim zt_\alpha$, then $hw = h'w$ and $w \in \overline{zT}$. Since z is an almost periodic point of (X, T), $\overline{zT}$ is minimal. Thus we obtain $y = hz = h'z = x$. Therefore ϕ is distal.

Using 5.5 in [3] and the distality lifts by distal homomorphisms, Proposition 6.6 of [3] is a corollary of Theorem 1 as follows:

COROLLARY 1. *Let (H, X, T) be a bitransformation group such that X/H is Hausdorff and $(X/H, T)$ is distal. Then (X, T) is distal.*

REMARK. Let $\phi: (X, T) \longrightarrow (Y, T)$ be a distal homomorphism. Then in general there does not exist a topological group H such that (H, X, T) is a bitransformation group and ϕ induces an isomorphism of $(X/H, T)$ onto (Y, T). Because, if we consider a minimal distal transformation group (X, T)

such that (X, T) is not regular minimal, (we will give an example of such a transformation group), and $\phi : (X, T) \longrightarrow (Y, T)$ is the onto homomorphism, where Y is the singleton space, then ϕ is distal. Suppose there exists a topological group H such that (H, X, T) is a bitransformation group and ϕ induces an isomorphism of $(X/H, T)$ onto (Y, T). Then (X, T) is regular minimal. Because, let x, x' be any two points of (X, T), then x' belongs to Hx. Therefore there exists an element $h \in H$ such that $hx = x'$. Since $h \in H$, h is an automorphism of (X, T) and $(hx, x') \in P(X, T)$. Therefore (X, T) is regular minimal. This contradicts our hypothesis on (X, T).

EXAMPLE. Let $X = \{1, 2, 3\}$ be a discrete topological space and $T = S_3$ be the permutation group of X. Then (X, T) is a minimal distal transformation group. But (X, T) is not regular minimal since (X, T) is not isomorphic to $(E(X), T)$ (see Theorem 3 in [2]).

Ellis has shown [3] that the map $\theta_x : p \longrightarrow xp\,(=p(x))$ of $E(X)$ into X is a homomorphism and if $\phi : (X, T) \longrightarrow (Y, T)$ is an epimorphism, there exists a unique epimorphism $\theta : (E(X), T) \longrightarrow (E(Y), T)$ such that the diagram

$$\begin{array}{ccc} E(X) & \xrightarrow{\ \ \theta\ \ } & E(Y) \\ \Big\downarrow \theta_x & & \Big\downarrow \theta_{\phi(x)} \\ X & \xrightarrow{\ \ \phi\ \ } & Y \end{array} \qquad \text{commutes } (x \in X).$$

Let $\phi : (X, T) \longrightarrow (Y, T)$ be an epimorphism, then which "dynamical" properties of (Y, T) lift to (X, T)? In general very little can be said. (Set Y equal to a one point space. Then any transformation group (X, T) can be mapped homomorphically onto (Y, T)).

In 6.2 [3], Ellis has shown that pointwise almost periodicity lifts from $(X/H, T)$ to (X, T). In Theorem 1, we have shown that the natural map $(X, T) \longrightarrow (X/H, T)$ is a distal homomorphism. The following theorem is a generalization of Ellis' result.

THEOREM 2. *Let* $\phi : (X, T) \longrightarrow (Y, T)$ *be distal and* (Y, T) *be pointwise almost periodic. Then* (X, T) *is pointwise almost periodic.*

Proof. Let x be any point of (X, T). Then $y = \phi(x)$ is an almost periodic point since (Y, T) is pointwise almost periodic. Therefore there exists an almost periodic point z of (X, T) with $\phi(z) = y$ by Proposition 6.1 in [3].

Let I be a minimal right ideal of $E(X)$ of the transformation group (X, T). Since z is an almost periodic point, there exists an idempotent $u \in I$

such that $zu=z$ (see 3.7 [3]). Since $y=\phi(z)=\phi(zu)=\phi(z)\theta(u)=y\theta(u)$, we have $\phi(xu)=\phi(x)\theta(u)=y\theta(u)=y$. Thus xu, x belong to $\phi^{-1}(y)$ and $(x, xu)\in P(X, T)$. Since ϕ is distal, we have $x=xu$. Therefore (X, T) is pointwise almost periodic (also see 3.7 [3]).

Ellis has shown that the product of a minimal distal transformation group and a minimal one is pointwise almost periodic. (See p. 119 in [3]) He proved this result with the use of T-subalgebras. We show that this is a corollary of Theorem 2.

COROLLARY 2. *The product of a distal transformation group and a pointwise almost periodic one is pointwise almost periodic.*

Proof. Let (X, T) be distal and (Y, T) be pointwise almost periodic. Let $\pi_Y : (X\times Y, T)\longrightarrow(Y, T)$ be the projection. Then π_Y is an epimorphism and π_Y is a distal homomorphism. Therefore $(X\times Y, T)$ is pointwise almost periodic by Theorem 2.

The following theorem is a generalization of (1) of Proposition 6.4 in [3].

THEOREM 3. *Let $\phi : (X, T)\longrightarrow(Y, T)$ be a distal homomorphism. If (Y, T) is minimal and $y\in Y$, then $\{\overline{xT}\mid x\in\phi^{-1}(y)\}$ is a partition of X.*

Proof. Let z be any point of (X, T). Then $\phi(z)\in(Y, T)$. Since (Y, T) is minimal and $\phi(z)$, y are elments of (Y, T), there exists a net (t_α) in T such that $y=\lim\phi(z)t_\alpha$. We may assume that $\lim zt_\alpha$ exists. Since $\phi(\lim zt_\alpha)=\lim\phi(z)t_\alpha=y$, we have $\lim zt_\alpha\in\phi^{-1}(y)$. Therefore $\lim zt_\alpha\in\overline{xT}$ for some $x\in\phi^{-1}(y)$. Since $\overline{xT}$ is a minimal set by Theorem 2, z belongs to $\overline{xT}$. Therefore $\{\overline{xT}\mid x\in\phi^{-1}(y)\}$ is a partition of X by Theorem 2.

THEOREM 4. *Let $\phi : (X, T)\longrightarrow(Y, T)$ be a distal epimorphism. Then $\theta : (E(X), T)\longrightarrow(E(Y), T)$ is also distal. The converse does not hold.*

Proof. Let p and q be elements in $E(X)$ such that $\theta(p)=\theta(q)$ and $(p, q)\in P(E(X), T)$. Then there exists a net (t_α) in T such that $\lim pt_\alpha=\lim qt_\alpha$. Let x be any element of (X, T). Then θ_x is a homomorphism from $(E(X), T)$ into (X, T). Thus $\theta_x(\lim pt_\alpha)=\lim\theta_x(p)t_\alpha=\lim(xp)t_\alpha$ and $\theta_x(\lim qt_\alpha)=\lim(xq)t_\alpha$. Therefore we have $\lim(xp)t_\alpha=\lim(xq)t_\alpha$. Thus we get $(xp, xq)\in P(X, T)$.

Since $\phi(xp)=\phi(x)\theta(q)=\phi(x)\theta(q)=\phi(xq)$ and ϕ is distal, we obtain $xp=xq$. Since x is any point of (X, T), we have $p=q$. Therefore ϕ is distal.

Before proving the converse of the theorem does not hold, we need the following lemma:

LEMMA. *Let* $\phi : (X \times X, T) \longrightarrow (Y, T)$ *be an epimorphism. Then* (X, T) *is distal if and only if* ϕ *is distal and* (Y, T) *is pointwise almost periodic.*

Proof. Only if: This is trivial.

If: By Theorem 2, $(X \times X, T)$ is pointwise almost periodic. Thus (X, T) is distal by (5.9) in [3].

Let (X, T) be a pointwise almost periodic (or minimal) transformation group which is not distal. If we consider the following diagram:

$$\begin{array}{ccc} E(X \times X) & \xrightarrow{\ \theta\ } & E(X) \\ \downarrow & \pi = \text{projection} & \downarrow \\ X \times X & \longrightarrow & X, \end{array}$$

then by Corollary 3.10 in [3], θ is the identity homomorphism. Thus θ is distal. If the converse of Theorem 3 holds, π is also distal. Since (X, T) is pointwise almost periodic and π is distal, we have (X, T) is distal by Lemma. This contradicts our hypothesis. Therefore the converse of Theorem 3 does not hold.

References

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