J. Korean Math. Soc.
Vol. 14, No. 2, 1978.

# ON THE SELF DUAL LOCALLY COMPACT ABELIAN GROUPS WITH COMPACT RADICAL

BY KWANG CHUL HA

1. It is the object of this note to describe a certain structure of self dual locally compact Abelian groups with compact radical. The main theorem is the following: Let $G$ be a locally compact Abelian group with compact radical. Then $G$ is self dual if and only if $G$ is the direct product $R^n \times D \times D^* \times H$, where $R^n$ is the real $n$-dimensional Euclidean space $(n \geqslant 0)$ and $D$ is a discrete torsion-free group and $D^*$, its dual, is compact connected, and $H$ is a finite discrete group. The proof is sketched in Section 2 and the preliminary theorems and notations used to derive the theorem 2.2 form Section 1.

DEFINITION 1.1. Let $G$ be a locally compact Hausdorff Abelian group with character group $G^\wedge$. (Hereafter we shall denote such groups $G$ as *LCA* groups.) $G$ is called *self dual* if there is a topological isomorphism $\alpha : G \to G^\wedge$ from $G$ onto $G^\wedge$.

Some examples of self dual groups are well known in the literature, but the structure of such groups is an open problem (see page 423 of [2] and [4].) In this note we investigate the structure of those self dual *LCA* groups which have compact radicals. We state some theorems required for proving the main theorem.

THEOREM 1.2. [3] *The radical of the Cartesian product of topological groups is the Cartesian product of the radicals of the original groups.*

THEOREM 1.3. [2] *Let $H$ be a compact Abelian group that is a pure subgroup of some Abelian group $G$. Then $G$ is algebraically isomorphic with $H \times (G/H)$.*

THEOREM 1.4. [2] *A locally compact Abelian group $G$ is topologically isomorphic with $R^n \times G_0$, where $G_0$ is a locally compact Abelian group containing a compact open subgroup.*

2. The radical of topological Abelian groups is investigated to a certain

Received by the editors Oct. 20, 1977.

extent in [1], [3], and [5]. If $G$ is an $LCA$ group and $T(G)$ is its radical, then $T(G)$ coincides with the set of all compact elements of $G$ and $T(G)$ is a closed subgroup of $G$. In particular any compact subgroup of $G$ is a subgroup of $T(G)$.

LEMMA 2.1. [3] *Let $G$ be an $LCA$ group, $T(G)$ its radical. If $ny \in T(G)$ for some positive integer $n$, then $y \in T(G)$. Hence $G/T(G)$ is torsion-free.*

*Proof*: Suppose for a positive integer $n$ and $y \in G$, $ny \in T(G)$. Since $ny \in T(G)$, $(ny)^-$, the closure of the group generated by $ny$, is compact. Hence $(y)^- = \bigcup_{t=0}^{n-1} (ty + (ny)^-)$ is compact.

We note that $(G/T(G))^\wedge$ is isomorphic to $K(G^\wedge)$, the component of the identity of $G^\wedge$, and $G/T(G)$ is torsion-free, which implies $K(G^\wedge)$ is divisible, or more generally the component of the identity of an $LCA$ group is divisible.

THEOREM 2.2. *Let $G$ be an $LCA$ group with compact radical. Then $G$ is self dual if and only if $G$ is the direct product $R^n \times D \times D^* \times H$, where $R^n$ is the real $n$-dimensional Euclidean space ($n \geqslant 0$) and $D$ is a discrete torsion-free group and $D^*$, its dual, is compact connected (divisible), and $H$ is a finite discrete group.*

*Proof*: Since it is obviously sufficient by Theorem 1.2, we work in only one direction.

By Theorem 1.4 $G \cong R^n \times G_0$, where $G_0$ contains a compact open subgroup. We know $G_0$ is self dual, and $T(G_0) \cong T(G_0) \times T(R_n) \cong T(G)$ is compact, by hypothesis. Since the component of $G_0^\wedge$, $K(G_0^\wedge)$ is the annihilator of $T(G_0)$, and since $T(G_0)$ is compact in $G_0$, $K(G_0^\wedge)$ is open in $G_0^\wedge$. Hence the component of the self dual group $K(G_0)$, is open which implies $G_0 \cong G_0/K(G_0) \times K(G_0)$ topologically. Now since $T(G_0)/K(G_0)$ is pure in $G_0/K(G_0)$ by 2.1, $G_0/K(G_0) \cong G_0/T(G_0) \times T(G_0)/K(G_0)$. Hence $G \cong R^n \times K(G_0) \times G_0/T(G_0) \times T(G_0)/K(G_0)$. Let $D = G_0/T(G_0)$ and $H = T(G_0)/K(G_0)$, then $D$ is a discrete torsion-free group, its character group $D^* = K(G_0)$ is compact connected (divisible), and $H$ is finite.

I am indebted to K. A. Ross for an improvement of a portion of the proof of theorem 2.2.

## References

[1] J. E. Diem and F. B. Wright, *Real Characters and the Radical of an Abelian Group*. Trans. Amer. Math. Society **129**, 1967, 517–529.

] E. Hewitt and K. A. Ross, *Abstract harmonic analysis I*, Springer-Verlag, Berlin, 1963.

] K. C. Ha, *The radical of topological Abelian groups*, to appear.

] M. Rajagopalan and T. Soundarajan, *On self dual LCA groups*, Bull. Amer. Math. Soc., Vol. **73**, 1967, 985-986.

] F. B. Wright, *Topological Abelian groups*, Amer. J. Math. **79**, 1957, 477-496.

Illinois State University