韓國數學敎育學會誌〈數學敎育〉
1978. 6. Vol. XVI No. 2

Journal of the Korea Society of Mathematical Education
Jun. 1978. Vol. XVI No. 2

On The Reflection And Coreflection

By Bae-Hun Park

Gyengsang National University, Jinju, Korea

Abstract: It is shown that a map having an extension to an open map between the Alexandroff base compactifications of its domain and range has a unique such extension. J. S. Wasileski has introduced the Alexandroff base compactifications of Hausdorff spaces endowed with Alexandroff bases. We introduce a definition of morphism between such spaces to obtain a category which we denote by ABC. We prove that the Alexandroff base compactification on objects can be extended to a functor on ABC and that the compact objects give an epireflective subcategory of ABC.

For each topological space X there exists a completely regular space αX and a surjective continuous function $\alpha_X : X \longrightarrow \alpha X$ such that for each completely regular space Z and $g \in C(X, Z)$ there exists a unique $\bar{g} \in C(\alpha X, Z)$ with $g = \bar{g} \circ \alpha_X$.

Such a pair $(\alpha_X, \alpha X)$ is called a completely regularization of X. Let TOP be the category of topological spaces and continuous functions and let CREG be the category of completely regular spaces and continuous functions. The functor $\alpha : \text{TOP} \longrightarrow \text{CREG}$ is a completely regular reflection functor.

For each topological space X there exists a compact Hausdorff space βX and a dense continuous function $\beta_X : X \longrightarrow \beta X$ such that for each compact Hausdorff space K and $g \in C(X, K)$ there exists a unique $\bar{g} \in C(\beta X, K)$ with $g = \bar{g} \circ \beta_X$. Such a pair $(\beta_X, \beta X)$ is called a Stone-Cech compactification of X. Let COMPT_2 be the category of compact Hausdorff spaces and continuous functions. The functor $\beta : \text{TOP} \longrightarrow \text{COMPT}_2$ is a compact reflection functor.

For each topological space X there exists a realcompact space $\mathcal{V}X$ and a dense continuous function $\mathcal{V}_X : X \longrightarrow \mathcal{V}X$ such that for each realcompact space Z and $g \in C(X, Z)$ there exists a unique $\bar{g} \in C(\mathcal{V}X, Z)$ with $g = \bar{g} \circ \mathcal{V}_X$. Such a pair $(\mathcal{V}_X, \mathcal{V}X)$ is called a Hewitt's realcompactification of X. Let RCOM be the category of realcompact spaces and continuous functions. The functor $\mathcal{V} : \text{TOP} \longrightarrow \text{RCOM}$ is a realcompact reflection functor.

In [2], D. Harris established the existence of a category of spaces and maps on which the Wallman compactification is an epirefiective functor. H. L. Bentley and S. A. Naimpally [1] generalized the result of Harris concerning the functorial properties of the Wallman compactification of a T_1-space. J. S. Wasileski [5] constructed a new compactification called Alexandroff base compactification.

In order to fix our notations and for the sake of convenience, we begin with recalling reflection and Alexandroff base compactification.

1. Reflection

Definition 1. 1. Let $\mathcal{A}$ be a subcategory of $\mathcal{B}$ with embedding functor $E : \mathcal{A} \longrightarrow \mathcal{B}$:

(1) An E-universal map (r_B, A_B) for a $\mathcal{B}$-object B is called an *$\mathcal{A}$-reflection of* B.

(2) $\mathcal{A}$ is called *reflective in* $\mathcal{B}$ or a *reflective subcategory of* $\mathcal{B}$ if and only if there exists an $\mathcal{A}$ -reflection for each $\mathcal{B}$ -object; i. e., if and only if E has a left adjoint, $R : \mathcal{B} \longrightarrow \mathcal{A}$. In this case, R is called a reflector for $\mathcal{A}$.

(3) If $\mathcal{E}$ is a class of $\mathcal{B}$-morphisms, then $\mathcal{A}$ is called *$\mathcal{E}$-reflective in* $\mathcal{B}$ provided that for each $\mathcal{B}$-object B there exists an $\mathcal{A}$ -reflection (r_B, A_B) such that each $r_B \in \mathcal{E}$. For the case that $\mathcal{E}$ is the class of all epimorphisms [resp. monomorphisms; extremal epimorphisms] of $\mathcal{B}$ we say that $\mathcal{A}$ is epireflective [resp. monoreflective; (extremal epi)-reflective] in $\mathcal{B}$.

Dual notions: (i. e., with respect to E co-universal maps and right adjoints to E) $\mathcal{A}$-coreflection of B; coreflective in $\mathcal{B}$ (or a coreflective subcategory of $\mathcal{B}$); coreflector for $\mathcal{A}$; $\mathcal{E}$-coreflective in $\mathcal{B}$, especially monocoreflective [resp. epicoreflective, (extremal mono)-coreflective] in $\mathcal{B}$.

Theorem 1. 2. *If $\mathcal{B}$ is complete, well-powered, and co-(well-powered), then the followings are equivalent:*

(1) *$\mathcal{A}$ is epirefledtive in $\mathcal{B}$.*

(2) *$\mathcal{A}$ is strongly closed under the formation of I-limits in $\mathcal{B}$ for each small category I.*

(3) *$\mathcal{A}$ is strongly closed under the formation of products and pullbacks in $\mathcal{B}$.*

(4) *$\mathcal{A}$ is strongly closed under the formation of products and inverse images in $\mathcal{B}$.*

(5) *$\mathcal{A}$ is strongly closed under the formation of products and finite intersections in $\mathcal{B}$.*

(6) *$\mathcal{A}$ is strongly closed under the formation of products and intersections in $\mathcal{B}$.*

Theorem 1. 3. *Let $\mathcal{A}$ be a full reflective subcategory of $\mathcal{B}$ with embedding $E: \mathcal{A} \longrightarrow \mathcal{B}$ and reflector $R: \mathcal{B} \longrightarrow \mathcal{A}$. Then*

(1) $ROE \approx 1_{\mathcal{A}}$, *and*

(2) $(E \circ R) \circ (E \circ R) \approx E \circ R$.

2. Alexandroff base compactification

In what follows, all spaces will be assumed Hausdorff.

Definition 2. 1. Let $\mathcal{B}$ be a set of subsets of a space X. We define a relation $<$ (rl $\mathcal{B}$) on subsets of X as follows: $A_1 < A_2$ (rl $\mathcal{B}$) means there exists $G_1, G_2 \in \mathcal{B}$ so that $A_1 \subset G_1$, $X\text{-}A_2 \subset G_2$ and $G_1 \cap G_2 = \phi$.

If $A_1 < A_2$ (rl$\mathscr{B}$) we say "A_1 is well-inside A_2 relative to $\mathscr{B}$". We will drop the phrase (rl$\mathscr{B}$) when no confusion can occur.

Definition 2.2. $\mathscr{B}$ is an *Alexandroff base* for X if and only if $\mathscr{B}$ is a base for the open sets of X satisfying:

(1) $\mathscr{B}$ is closed under finite unions and intersections.

(2) $p \in G \in \mathscr{B} \Longrightarrow$ there is an $H \in \mathscr{B}$ satisfying $p \in H < G$ (rl$\mathscr{B}$).

(3) $\mathscr{B}$ is densely ordered by the "well-inside" relations it defines.

As an example of Alexandroff bases one can consider the open sets of a normal space. Each completely regular space has an Alexandroff base.

Definition 2.3. If $\mathscr{B}$ is an Alexandroff base for X then δ, a non-empty collection of non-empty members of $\mathscr{B}$, is called a *$\mathscr{B}$-filter* (on X) provided δ satisfies:

(1) $G_1, G_2 \in \delta \Longrightarrow G_1 \cap G_2 \in \delta$;

(2) $G_1 \in \delta$ and $G_2 \in \mathscr{B}$ with $G_1 \subset G_2 \Longrightarrow G_2 \in \delta$;

(3) $G_1 \in \delta \Longrightarrow$ there exists $G_2 \in \delta$ with $G_2 < G_1$ (rl$\mathscr{B}$).

A maximal $\mathscr{B}$-filter will be called a cluster.

Lemma 2.4 *Every $\mathscr{B}$ -filter is contained in a cluster.*

Lemma 2.5. *If $\mathscr{B}$ is an Alexandroff base for X and $x \in X$, then*

$$x^* = \{G \in \mathscr{B} : x \in G\}$$

is a cluster on X.

Lemma 2.6. *If δ is a cluster and β is a $\mathscr{B}$-filter such that $\beta \neq \delta$, then $\beta \subset \delta$ or β and δ contain disjoint members.*

For a given Alexandroff base $\mathscr{B}$ we denote by $\alpha_{\mathscr{B}}X$ the set of clusters defined by $\mathscr{B}$; whenever no confusion can result, we simply use the symbol αX. We define, for each $G \in \mathscr{B}$, the set

$$G^* = \{\delta \in \alpha X : G \in \delta\}.$$

The family $\{G^* : G \in \mathscr{B}\}$ is a base for a topology on αX.

Lemma 2.7. *If* $\phi \neq G < H$, then $\overline{G^*} \subset H^* \subset int(\overline{H^*})$ *where the closure and interior operators are those in αX.*

Theorem 2.8. *αX is a Hausdorff compact space.*

For each $x \in X$, let $\alpha_X(x) = x^*$. Then $\alpha_X : X \longrightarrow \alpha X$ is a dense embedding of X in αX and $\alpha_X[G] = G^* \cap \alpha_X[X]$.

3. The Alexandroff base compactification is an epireflection

A space is a Hausdorff space and a map is a continuous function.

The class of objects of category ABC will be the class of spaces and the morphisms wil be the class of maps $f : X \longrightarrow Y$ such that there is a open map $\alpha f : \alpha X \longrightarrow \alpha Y$ with $\alpha f \alpha_X = \alpha_Y f$, where α_X, α_Y are the inclusions of X, Y into their Alexandroff base compactification

αX, αY. Any such an open map αf is called an α-extension of f.

Lemma 3.1. *For each $X \in ABC$, the maps I_X and α_X are morphisms.*

Lemma 3.2. *If f is a morphism and αf is an α-extension of f then αf is a morphism.*

Since the composition of open maps is an open map it follows that the composition of m or-phisms is a morphism.

To show that the Alexandroff base compactification is an epireflection on this category we need only show that each morphism has a unique α-extension.

Lemma 3.3. *If $f: X \longrightarrow Y$ is a morphism and $\alpha f: \alpha X \longrightarrow \alpha Y$ is an α-extension of f then for any open $U \subset \mathcal{B}_Y$ we have $U \in \alpha f(\delta)$ if and only if there is $V \in \delta$ with $int_Y f(V) \subset U$, where $\mathcal{B}_Y$ is Alexandroff base for Y.*

Proof: Suppose $V \in \delta$. Then $\delta \in V^* = int_{\alpha X}\alpha_X(V)$ and $\alpha f(\delta) \in \alpha f(int_{\alpha X}\alpha_X(V)) \subset int_{\alpha Y}\alpha f(\alpha_X(V))$ since αf is open map. By the definition of αf, $int_{\alpha Y}\alpha f(\alpha_X(v)) = int_{\alpha Y}\alpha_Y f(V)$. Thus $\alpha f(\delta) \in int_{\alpha Y}\alpha_Y f(V)$. It follows that $int_Y f(V) \in \alpha f(\delta)$. If $int_Y f(V) \subset U$ for some $V \in \delta$ and open $U \subset Y$ then $B \subset U$ for some $B \in \alpha f(\delta)$ and $U \in \alpha f(\delta)$.

Conversely suppose $U \in \alpha f(\delta)$ and open $U \subset Y$. then $\alpha f(\delta) \in U^*$ and so $\alpha f^{-1}(U^*)$ is an open neighborhood of δ. Hence there exists a $W \in \delta$ with $W^* \subset \alpha f^{-1}(U^*)$. By the definition of δ, since $W \in \delta$, there exists a $C \in \delta$ with $C < W$ (r1$\mathcal{B}$). Now $int_{\alpha X}\alpha_X(C) = C^* \subset W^*$, and since αf is open map we have $int_{\alpha Y}\alpha_Y f(C) = int_{\alpha Y}\alpha f(\alpha_X C) = int_{\alpha Y}\alpha f(int_{\alpha X}\alpha_X C) = \alpha f(C^*) \subset \alpha f(W^*) \subset U^*$. Therefore $int_Y f(C) \subset U$. Thus if $U \in \alpha f(\delta)$ there is $C \in \delta$ with $int_Y f(C) \subset U$.

In accordance with the lemma 3.3, the point $\alpha f(\delta)$ is entirely determined by the map f, hence the following holds:

Corollary 3.4. *Each morphism has a unique α-extension.*

Corollary 3.5. *Each α_X is an epimorphism.*

Proof: Suppose αf and αg are morphisms with $\alpha f\alpha_X = \alpha g\alpha_X$. Then αf and αg have α-extensions m and n. Since $n\alpha_X = \alpha_Y\alpha f\alpha_X = \alpha_Y\alpha g\alpha_X = m\alpha_X$. By corollary 3.4, n=m and therefore $\alpha_Y\alpha g = n = m = \alpha_Y\alpha f$. Thus $\alpha g = \alpha f$.

Theorem 3.6. *The category of open maps and compact spaces is an epireflective subcategory of the category ABC.*

References

1. H.L. Bentley and S.A. Naimpally(1974), Wallman T_1-compactifications as epireflec tions, *General topology and its applications* 4 29-41.
2. D. Harris(1972), The Wallman compactification is an epireflection, *Proc. of the A mer. Math. Soc.* vol. 31, No. 1, Jan.
3. H. Herrich, (1973) *Topologische Reflexionen und Coreflexionen*, Springer, 1968.
4. _______, *Category Theory*, Allyn and Bacon Inc.,
5. J.S. Wasileski, (1974), Compactifications, *Can. J. Math.*, Vol. XXVI, No. 2. 365-371.

국 문 초 록

$(\alpha_X, \alpha X)$와 $(\alpha_Y, \alpha Y)$를 T_2 공간 X와 Y의 Alexandroff base Compactification이라 할 때 $\alpha f \cdot \alpha_X = \alpha_Y f$를 만족하는 open이고 연속인 함수 $\alpha f : \alpha X \longrightarrow \alpha Y$가 존재하는 연속함수 $f : X \longrightarrow Y$는 유일한 α-extension αf를 가지며 Category ABC를 T_2 공간과 위와 같은 연속함수 f들의 Category라고 할 때 open이고 연속인 함수와 Compact space들의 Category는 Category ABC의 epireflective subcategory임을 밝혔다.