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Strong Topologies On Generalized Inner Product Spaces

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1. Introduction

An inner product space is a vector space on which an inner product (x,y) is defined. When the vector space is complex, we adopt the convention that (x,y) is anti-linear with respect to the first argument, and consequently linear with respect to the second argument.

A vector space $\mathcal{L}$ is a generalized inner product space if and only if:

1. There is a subspace $\mathfrak{N}$ of $\mathcal{L}$ which is an inner product space;

2. There is a set $\mathcal{A}$ of linear operators on $\mathcal{L}$ which is adequate with respect to $\mathfrak{N}$, i.e., it has the following properties:

(a) Each element of $\mathcal{A}$ maps $\mathcal{L}$ into $\mathfrak{N}$, i.e. $\mathcal{A}\mathcal{L}\subset\mathfrak{N}$;

(b) The relation $Ax=0$ is satisfied for all $A\in\mathcal{A}$ only by $x=0$.

We denote such a generalized inner product space by the triple $(\mathcal{L},\mathcal{A},\mathfrak{N})$. Clearly, every inner product space is also a generalized inner product space in a trivial sense, i.e. $\mathfrak{N}=\mathcal{L}$ and $\mathcal{A}=\{1\}$ where 1 denotes the identity operator on $\mathcal{L}$. A non-trivial example is the following.

Example. Take $\mathcal{L}$ is the family of all real continuous functions on the real line. Choose $\mathfrak{N}$ to consist of all square integrable functions in $\mathcal{L}$ and adopt the inner product in $\mathfrak{N}$ to be

$$(x,y)=\int_{-\infty}^{+\infty}x(t)y(t)dt.$$

Take $\mathcal{A}$ to be the family of all projectors $E(I)$,

$$(E(I)x)(t)=\chi_{I}(t)x(t)$$

($\chi_{s}(t)$ denotes the characteristic function of the set S) corresponding to all the finite non-degenerate intervals. It is straightforward to check that the present $(\mathcal{L},\mathcal{A},\mathfrak{N})$ is a generalized inner product space.

In this paper we study the normability and metrizability of generalized inner product spaces with strong topology.

2. Main theorems

There are obviously many convenient ways to introduce a topology in a generalized inner product space in order to obtain a topological vector space. We shall introduce in generalized inner product spaces strong topologies by constructing neighbourhood bases of some point

$x \in \mathcal{L}$ from sets of the form

$$V(x; A_1, \cdots, A_n; \varepsilon) = \{y : \| A_1(y-x) \| < \varepsilon, \cdots, \| A_n(y-x) \| < \varepsilon, \ y \in \mathcal{L}\}$$

for all $\varepsilon > 0$, $A_1, \cdots, A_n \in \mathcal{A}$ and $n = 1, 2, \cdots$.

In the strong topology on the generalized inner product space $(\mathcal{L}, \mathcal{A}, \mathcal{N})$, defined as above, the space $\mathcal{L}$ is a locally convex Hausdorff topological vector space (cf. 2, Chapter 2, §4).

If the topology of topological vector space can be defined a norm, we say that it is normable.

Theorem 1. *A generalized inner product space $\mathcal{L}$ with strong topology is normable if $\mathcal{A}$ is a finite family.*

Proof. Let $\mathcal{A} = \{A_1, \cdots, A_n\}$ be a finite family. Then the function q defined by

$$q(x) = \max_{1 \leqq i \leqq n} \| A_i(x) \|$$

is a semi-norm on $\mathcal{L}$. Since this space is a Hausdorff topological vector space, q is a norm on $\mathcal{L}$. We have that

$$\{x \mid q(x) < \varepsilon\} = \{x \mid \| A_i(x) \| < \varepsilon, \ 1 \leqq i \leqq n\}$$

which shows that the family $\{x \mid q(x) < \varepsilon\}$ is a neighbourhood basis of the origin, hence $\mathcal{L}$ is normable with the strong topology.

We say that a topological space X is metrizable if there exists a metric on X such that the topology defined by it coincides with the topology of X. Clearly, a metrizable space is always Hausdorff, and each point possesses a countable bases. In a topological vector space the converse is also true (cf. 2, Chapter 2, §6).

Theorem 2. *A generalized inner product space $(\mathcal{L}, \mathcal{A}, \mathcal{N})$ with strong topology is metrizable if there is a countable subset $\mathcal{B} = \{A_1, A_2, \cdots\}$ of $\mathcal{A}$ which has the property that for any $A \in \mathcal{A}$ there is an n such that*

$$\max_{1 \leqq i \leqq n} \| A_i(x) \| \geqq \| A(x) \|$$

for all $x \in \mathcal{L}$.

Proof. Let τ denotes the strong topology generated by the family $\mathcal{A}$ and τ_1 denotes the strong topology generated by the family $\mathcal{B}$. Clearly, τ is finer than τ_1. On the other hand it follows from

$$\max_{1 \leqq i \leqq n} \| A_i(x) \| \geqq \| A(x) \|$$

that the identity map $(\mathcal{L}, \tau_1) \rightarrow (\mathcal{L}, \tau)$ is continuous and thus τ_1 is finer than τ. Therefore τ_1 coincides with τ. Since the space $\mathcal{L}$ with τ_1 is metrizable, $(\mathcal{L}, \mathcal{A}, \mathcal{N})$ is also metrizable.

References

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