韓國數學敎育學會誌〈數學敎育〉
1978. 12 Vol. XVII No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1978. Vol. XVII No. 1

A Note On a Class Of Normalized Analytic Functions

By Shin, Yong Soon

Ajou Institute of Technology, SuWon, Korea

Let D be the open unit disk in the complex plane, and let A be the set of all analytic functions of the form $f(z)=z+a_2z^2+a_3z^3+\cdots\cdots+a_nz^n+\cdots$ satisfying the following conditions.

1) $f'(z)\neq 0$ for every z in D.
2) a_2 is an integer.
3) $\arg f'(z)$ is bounded in D.

We define the operator

$$\oplus : A\times A \longrightarrow A$$

by $[\oplus(f,g)](z)=(f\oplus g)(z)=\int_0^z f'(w)g'(w)dw$

under this operation $<A, \oplus>$ is an abelian group.

On $<A, \oplus>$ we define real scalar multiplication

$$\otimes : R\times A \longrightarrow A$$

by $\quad r\otimes f=\int_0^z [f'(w)]^r\, dw,$

where $[f'(w)]^r$ is chosen so that $[f'(0)]^r=1$.

Then $A=<A, \oplus, \otimes>$ is a vector space over the field of real numbers.

We define a norm on A by

$$\| f \| = \frac{1}{\pi}\sup_{z_1, z_2\in D}\left|\arg\frac{f'(z_1)}{f'(z_2)}\right|.$$

Then $<A, \oplus, \otimes, \| \ \| >$ is complete with respect to $\| \ \|$, hence A is a Banach space.

Now we are ready to state our results.

Theorem 1. *If* $f\in A$ *and* $\| f \| <1$, *then* f *is schlicht in* D.

Proof Since $\| f \| <1$, there exists a real number r such that

$$\mathrm{Re}\; e^{ir}f'(z)>0 \quad \text{in D}.$$

Hence by the well known fact that $\mathrm{Re}\, f'(z)>0$ implies f is schlicht in D, $e^{ir}f(z)$ is schlicht in D. Thus f(z) is schlicht in D.

Theorem 2. *Let* $f(z)=z+a_nz^n+a_{n+1}z^{n+1}+\cdots$, $a_n\neq 0$, $n\geq 5$, *be analytic in* D *and* $f'(z)\neq 0$ *in* D. *Then if* $\| f \| \leq \dfrac{n}{\frac{1}{2}en+1.51}$, f *is schlicht in* D *and* $|a_n|\leq n$.

Proof: Since $\dfrac{n}{\frac{1}{2}en+1.51}<1$, f is schlicht in D by Theorem 1.

If $\|f\|=0$, then $f=z$. But by hypothesis $a_n \neq 0$, so $f(z) \neq z$. So we must have $\|f\| \neq 0$. Let r be any non zero real number, and consider the function $f_r(z)=r\otimes f=\int_0^z [f'(w)]^r dw$.

Then $\|f_r\|=|r|\,\|f\|$.

Hence f_r is schlicht for $|r| \leq \frac{1}{\|f\|}$ (1)

Now by direct calculation we have

$$\begin{aligned} f_r(z) &= \int_0^z [f'(w)]^r dw \\ &= \int_0^z (1+na_n w^{n-1}+\cdots)^r dw \\ &= \int_0^z (1+rna_n w^{n-1}+\cdots)\ dw \\ &= z+ra_n z^n+\cdots \end{aligned}$$

So if f_r is schlicht in D, we have, by Bozilevich's result,

$$|ra_n| < \frac{1}{2}en+1.51$$

Thus if $|r| \geq \left(\frac{1}{2}en+1.51\right)/|a_n|$ (2)

Then f_r is not schlicht.

Combining (1) and(2), we obtain

$$\frac{1}{\|f\|} \leq \frac{\frac{1}{2}en+1.51}{|a_n|}$$

Hence $|a_n| \leq \left(\frac{1}{2}en+1.51\right)\|f\| \leq n$.

This completes the proof of the theorem.

References

[1] W. K. Hayman (1967), *Research Problems in Function Theory* London Univ. Press Problem 6.15, p. 40.

[2] W. C. Royster (1965), On the univalence of certain integral, *Mich. Math. Jour.*, 12 pp. 385-387.

[3] J. A. Jenkins (1958), *Univalent functions and Conformal Mappings*, Springer Verlag, Berlin.

[4] L. A. Ahlfors (1966), *Complex Analysis*, McGraw-Hill.

[5] A. E. Taylor (1964), *Functional Analysis*, John Weley & Sons Inc. New York