韓國數學敎育學會誌(數學敎育)
1978. 12 Vol. XVII No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1978. Vol. XVII No. 1

Categorical Homotopy

By Yong-Seung Cho

Kyungpook National University, TaeGu, Korea

Let $\mathscr{C}$ be an arbitrary category, and $\mathscr{M}$ be any family of its morphisms. It is known that there is a category $\mathscr{C}/\mathscr{M}$ by Gabriel-Zisman. And the categoly $\mathscr{C}/\mathscr{M}$ has the same objects as $\mathscr{C}$, and a covariant functor $\eta:\mathscr{C}\longrightarrow\mathscr{C}/\mathscr{M}$ which is the identity on object, such that, $\eta(f)$ is invertible in $\mathscr{C}/\mathscr{M}$ for each $f\in\mathscr{M}$.

We will use each class $\mathscr{M}$ to determine a notion of homotopy in $\mathscr{C}$. In the category Top, a suitable choice of $\mathscr{M}$ determines the usual homotopy.

Definition. Let $\mathscr{C}$ be any category, and let $\mathscr{M}$ be any family of its morphisms. By a quotient category we shall mean a pair $(\mathscr{C}/\mathscr{M},\eta)$ where $\mathscr{C}/\mathscr{M}$ is a category with the same objects as $\mathscr{C}$ and $\eta:\mathscr{C}\longrightarrow\mathscr{C}/\mathscr{M}$ is a covariant functor that preserves objects, having the following two properties.

(1) If $\alpha\in\mathscr{M}$, then $\eta(\alpha)$ is invertible in $\mathscr{C}/\mathscr{M}$.

(2) If $T:\mathscr{C}\longrightarrow\mathscr{D}$ is any covariant functor to any category $\mathscr{D}$ such that $T(\alpha)$ is invertible for each $\alpha\in\mathscr{M}$, then there is a unique covariant functor $H:\mathscr{C}/\mathscr{M}\longrightarrow\mathscr{D}$ such that $T=H\circ\eta$.

Theorem 1. *Let $\mathscr{C}$ be any category, and let $\mathscr{M}$ be any family of its morphisms. Then a quotient category $(\mathscr{C}/\mathscr{M},\eta)$ exists.*

Definition. Let $R:\mathscr{C}\longrightarrow\mathscr{D}$ be a covariant functor. For $f,g\in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ f and g are *R-homotophic* (written $f\overset{R}{\sim}g$) if $R(f)=R(g)$.

Definition. A morphism $f:X\longrightarrow Y$ in a category $\mathscr{C}$ is called *constant* if for each object $Z\in \mathrm{Ob}\mathscr{C}$, and for each pair of morphisms $g,h:Z\longrightarrow X$ it follows that $fg=fh$.

Definition: A category $\mathscr{C}$ is a *category with constant morphisms* if for each pair (X,Y) in $\mathscr{C}$, there is a constant morphism in $\mathrm{Hom}(X,Y)$.

Now we have some consequences.

Theorem 2. *Let* $R:\mathscr{C}\longrightarrow\mathscr{D}$ *be a covariant functor. Then*

(1) R-*homotopy is an equivalence relation in each* $\mathrm{Hom}_{\mathscr{C}}(X,Y)$.

(2) *If* $f_0\overset{R}{\sim}f_1$ *in* $\mathrm{Hom}_{\mathscr{C}}(X,Y)$, *for each* $h\in\mathrm{Hom}_{\mathscr{C}}(Z,X)$, $g\in\mathrm{Hom}_{\mathscr{C}}(Y,W)$, *then* $f_0h\overset{R}{\sim}f_1h$, $gf_0\overset{R}{\sim}gf_1$.

(3) *If* $f:X\longrightarrow Y$ *is isomorphism, then* f *is an* R-*homotophic equivalence.*

Theorem 3. *Let $\mathscr{C}$ be a category and $(\mathscr{C}/\mathscr{M},\eta)$ a quotient category.*

(1) *If* $\mathcal{M}$ *is the class of all monomorphisms in* $\mathcal{C}$ *and if* $N \xrightarrow{v} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ *is exact in, then* $f \overset{\eta}{\sim} g$.

(2) *If* $\mathcal{M}$ *is the class of all epimorphisms in* $\mathcal{C}$ *and if* $X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{u} Z$ *is exact in* $\mathcal{C}$*, then* $f \overset{\eta}{\sim} g$.

Proof It is obvious.

Theorem 4. *Let* $R : \mathcal{C} \longrightarrow \mathcal{D}$ *be a covariant functor, and* $\mathcal{M} = \{f \mid R(f) : \text{isomorphism in } \mathcal{D}\}$ *and let* $(\mathcal{C}/\mathcal{M}, \eta)$ *be a quotient category of* $\mathcal{C}$*. Then* η*-homotopy implies* R*-homotopy.*

Proof There is a unique covariant functor $H : \mathcal{C}/\mathcal{M} \longrightarrow \mathcal{D}$ such that $R = H \circ \eta$. Let $f \overset{\eta}{\sim} g$, then $R(f) = H\eta(f) = H\eta(g) = R(g)$.

Thus we have $f \overset{R}{\sim} g$.

Remark: In Theorem 3, if the covariant functor $H : \mathcal{C}/\mathcal{M} \longrightarrow \mathcal{D}$ is a faithful functor, then R-homotopy implies η-homotopy.

Theorem 5. *Let* $\mathcal{C}$ *be a category with constant morphisms, and let* $\mathcal{M}$ *be the class of all constant morphisms in* $\mathcal{C}$*, and* $(\mathcal{C}/\mathcal{M}, \eta)$ *be a quotient category of* $\mathcal{C}$*. Then*

(1) *If* $f, g : X \longrightarrow Y$ *are morphisms in* $\mathcal{C}$*, then* $f \overset{\eta}{\sim} g$.

(2) *If* $f : X \longrightarrow Y$ *is a morphism in* $\mathcal{C}$ *and* $\mathrm{Hom}_{\mathcal{C}}(Y, X) \neq \phi$*, then* f *is* η*-homotopy equivalence.*

Theorem 6. *Let* $\mathcal{C}$ *be category with constant morphisms and let* $\mathcal{M}$ *be the class of constant morphims. If* $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq \phi$, $\mathrm{Hom}_{\mathcal{C}}(Y, X) \neq \phi$*, then there is a monoid isomorphism* $\varphi : \mathrm{Hom}_{\mathcal{C}/m}(X, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}/m}(Y, Y)$ *in* $\mathcal{C}/\mathcal{M}$.

Proof Let $f \in \mathrm{Hom}\ (X, Y)$, $g \in \mathrm{Hom}\ (Y, X)$. Define $\varphi : \mathrm{Hom}_{\mathcal{C}/m}(X, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}/m}(Y, Y)$ by $\varphi(\alpha) = \eta(f)\alpha\eta(g)$ in $\mathcal{C}/\mathcal{M}$. Then φ is a monoid homomorphism.

Define a homomorphism $\psi : \mathrm{Hom}_{\mathcal{C}/m}(Y, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}/m}(X, X)$ by $\psi(\beta) = \eta(g)\beta\eta(f)$ for any $\beta \in \mathrm{Hom}_{\mathcal{C}/m}(Y, Y)$. Then $\psi\varphi(\alpha) = \alpha = 1(\alpha)$ for any $\alpha \in \mathrm{Hom}_{\mathcal{C}/m}(X, X)$.

Similarly $\varphi\psi = 1$. Thus we have $\varphi : \mathrm{Hom}_{\mathcal{C}/m}(X, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}/m}(Y, Y)$ is a monoid isomorphism.

Theorem 7. *In Top, let Top=*$\mathcal{C}$*. Let* $\mathcal{M}$ *be all inclusion functions* $i : A \longrightarrow X$ *and* A *be deformation retract of* X*, and* $(\mathcal{C}/\mathcal{M}, \eta)$ *be a quotient category. Then* η*-homotopy implies the usual homotopy.*

Proof Let $\sim$ be the usual homotopy relation in $\mathcal{C}$, and let $R : \mathcal{C} \longrightarrow \mathcal{C}/\sim$ be a covariant functor defined by $R(A) = A$, $R(f) = [f]$, where [f] is the usual homotopic class of f. Let $i : A \longrightarrow X$ be a inclusion and A be a deformation retract of X. Then there is $r : X \longrightarrow A$ a retraction and a continuous function $F : XxI \longrightarrow X$ such that $F(x, 0) = x = 1(x)$, $F(x, 1) = r(x) = ri(x)$.

Therefore $1_X \sim ri$, $ir \sim 1_A$. Thus R(i) is an isomorphism in $\mathcal{C}/\sim$. There is a unique covariant functor $H : \mathcal{C}/\mathcal{M} \longrightarrow \mathcal{C}/\sim$ such that $R = H \circ \eta$. If $f \overset{\eta}{\sim} g$, then $R(f) = H\eta(f) = H\eta(g) = R(g)$. Thus f and g are usual homotopic in Top.

Theorom 8. *In Top, let Top=*$\mathcal{C}$*. Let* $\mathcal{M}$ *be the class of all maps* $r : XxI \longrightarrow X$ *given by*

$r(x,t)=x$ *or* $\mathcal{M}$ *be the class of all inclusion* $i: A \longrightarrow X$ *and* A *is a zero set and a strong deformation retract of* X, *and let* $(\mathcal{C}/\mathcal{M}, \eta)$ *be a quotient category of* $\mathcal{C}$. *Then the* η*-homotopy coinsides with the usual homotopy.*

Remark: In theorem 8, there is a continous map $\phi: X \longrightarrow I$ with $A=\phi^{-1}(0)$.

References

[1] P. Gabriel and M. Zisman (1967), *Calculus of fractions and homotopy theory*, Springer-Verlag, New York.

[2] B. Mitchell (1965), *Theory of categories*, Academic press, New York.

[3] F. W. Bauer and J. Dugundji, *Categorical homotopy and fibrations.*

[4] S. Maclane (1971), *Categories for the Working Mathematician*, Springer-Verlag.