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Compare The Density Functions Between Two Kinds Of Random Variables

By Lee Sang Gone

Jeon Bug National University, Jeon Ju, Korea

1. Introduction

If $X_1, \cdots, X_n$ are mutually independent, gamma-distributed random variables with parameters $(\alpha_1, \lambda), \cdots, (\alpha_n, \lambda)$, respectively, then $X_1+\cdots+X_n$ is gamma-distributed with parameters $(\alpha_1+\cdots+\alpha_n, \lambda)$. In this case, if X_i is standard normally distributed, X_i^2 is gamma-distributed with parameters $\left(\frac{1}{2}, \frac{1}{2}\right)$ or, equivalently, $X_i^2/2$ is gamma-distributed with parameters $\left(\frac{1}{2}, 1\right)$.

Suppose that $X_1, \cdots, X_n$ are mutually independent, N(0, 1) random variables. then $\sum_{i=1}^{n} X_i^2$ is gamma-distributed with parameters $\alpha=\frac{1}{2}n$, $\lambda=\frac{1}{2}$.

An interesting example is the following.

Let (X_1, X_2) be a pair of independent, normally distributed random variables, each $N(0, \sigma^2)$ and XY-coordinates in the plane is discribed by (X_1, X_2). Then the distance of the origin (0, 0) is $(X_1^2+X_2^2)^{\frac{1}{2}}$. Since $(X_1^2+X_2^2)/2\sigma^2$ is exponentially distributed with parameter $\lambda=1$,

$$P((X_1^2+X_2^2)^{\frac{1}{2}}\leqq t)=P((X_1^2+X_2^2)/2\sigma^2\leqq t^2/2\sigma^2)=1-\exp(-t^2/2\sigma^2)$$

Suppose that there are there points in the plane; $P_1(X_1, Y_1)$, $P_2(X_2, Y_2)$, $P_3(X_3, Y_3)$. Of course, without loss of generality, we can consider that random variable $\frac{1}{2}((X_1^2+X_2^2+Y_1^2+Y_2^2)-((X_1-X_2)^2+(Y_1-Y_2)^2))$ is $X_1X_2+Y_1Y_2$. But the purpose of this article is to show that, between two kinds of random variables, there exists the difference in their integrating density functions.

2. Main Results

Theorem 1 *Let* $X_1, \cdots, X_n$ *be mutually independent, standard normally distributed random variables. then The jonit distribution of n independent variables* X_i *is*

$$dF=(2\pi)^{-\frac{1}{2}n}\exp\left(-\frac{1}{2}(X_1^2+\cdots+X_n^2)\right)dX_1\cdots dX_n.$$

Proof $P=(X_1X_2\cdots X_n\leqq t)=\iint\cdots\int (2\pi)^{-\frac{1}{2}n}\exp\left(-\frac{1}{2}\sum_{i=1}^{n}X_i^2\right)dX_1\cdots dX_n.$

Let us change the variate to $z, \theta_1, \theta_2, \cdots, \theta_{n-1}$,

where $X_1=\sqrt{z}\cos\theta_1\cos\theta_2\ \cos\theta_{n-1}$

$X_2=\sqrt{z}\cos\theta_1\cdots\cos\theta_{n-2}\sin\theta_{n-1}$

$\cdots$

$\cdots$

$X_n=\sqrt{z}\sin\theta_1$

and hence $X_1^2+X_2^2+\cdots\cdots+X_n^2=z$.

$$\frac{\partial(X_1,\ X_2,\cdots,\ X_n)}{\partial(z,\theta_1,\cdots,\theta_{n-1})}=z^{\frac{1}{2}(n-2)}f(\theta_1,\cdots,\ \theta_{n-1})$$

where $f(\theta_1,\cdots,\ \theta_{n-1})$ is a function of the θ's only. Hence

$$F(z)=(2\pi)^{-\frac{1}{2}n}\iint\cdots\int z^{\frac{1}{2}(n-2)}e^{-\frac{1}{2}z}f(\theta_1,\cdots,\theta_{n-1})d\theta_1\cdots d\theta_{n-1}dz$$

$(0\leqq\theta\leqq2\pi)$

$(0\leqq z\leqq\infty)$

$$F(z)=k\int_0^z z^{\frac{1}{2}(n-2)}e^{-\frac{1}{2}z}dz$$

where k is a constant given by

$$1=k\int_0^\infty z^{\frac{1}{2}(n-2)}e^{-\frac{1}{2}z}dz=k\cdot2^{\frac{1}{2}n}\cdot\Gamma\left(\frac{1}{2}n\right).$$

$$dF=2^{-\frac{1}{2}n}\left(\Gamma\left(\frac{1}{2}n\right)\right)^{-1}e^{-\frac{1}{2}z}z^{\frac{1}{2}(n-2)}dz.$$

Example; Projectiles are fired at the origin of an (X, Y) coordinate system, the mathematical model is that the point which is hit, (X, Y), consists of a pair of independent, $N(0, 1)$ random variable. Suppose that three projectiles are fired independently. Then P is the probability that the length P_1P_2, P_2P_3, P_3P_1 could form a triangle.

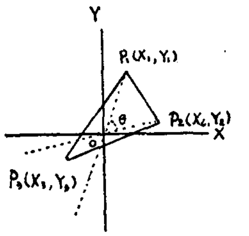

Fig. 1.

$$\cos\theta=\frac{X_1X_2+Y_1Y_2}{\sqrt{(X_1^2+Y_1^2)(X_2^2+Y_2^2)}},$$

$$P=\frac{\theta}{2\pi}$$

Theorem 2 *Let* (X, Y) *be a pair of independent, normally distributed random variables, each* $N(0, 1)$ *and* XY-*coordinate system is discribed by* (X_i, Y_i). *The density function of* $(X_1X_2+Y_1Y_2)$ *and* $(X_1^2+X_2^2+Y_1^2+Y_2^2)$ *is the same.* (*referring to fig.* 1)

Proof The joint integrating density function of X_1 and X_2 is

$$f(z_1)=\int_0^{z_1}\frac{1}{2}e^{-\frac{1}{2}z_1}dz_1$$

By the same way, the joint integrating density function of Y_1 and Y_2 is

$$f(z_2)=\int_0^{z_2}\frac{1}{2}e^{-\frac{1}{2}z_2}dz_2$$

The integrating density function of $X_1X_2+Y_1Y_2$ is

$$f(u)=\int_0^u \frac{1}{4}e^{-\frac{1}{2}(u-v)}e^{-\frac{1}{2}v}\,dv=\frac{1}{4}ue^{-\frac{1}{2}u} \qquad (u\geqq 0)$$

The integrating density function of $X_1^2+Y_1^2+X_2^2+Y_2^2$ is the chi-square distribution with 4 degrees of freedom(or exponential distribution).

$$g(x)=\frac{1}{4}xe^{-\frac{1}{2}x}$$

Therefore, the integrating density function of

$$X_1^2+Y_1^2+X_2^2+Y_2^2-((X_1-X_2)^2+(Y_1-Y_2)^2) \text{ is}$$

$$h(u)=\frac{1}{16}\int_0^u e^{-\frac{1}{4}(u+v)}ve^{-\frac{1}{2}v}\,dv=\frac{1}{9}e^{-\frac{1}{4}u}-\frac{1}{12}ue^{-1}-\frac{1}{9}e^{-1}$$

where the integrating density function of $(X_1-X_2)^2+(Y_1-Y_2)^2$ is $g(z)=\frac{1}{4}e^{-\frac{1}{4}z}$.

3. Summary and Conclusions

The important characterizations given by the Theorem are that the joint integrating density function of X_1X_2 and Y_1Y_2 are exponential distribution form. Without loss of generality, we find that there is the difference between the integrating density functions of random variables $(X_1^2+X_2^2+Y_1^2+Y_2^2)-((X_1-X_2)^2+(Y_1-Y_2)^2)$ and $X_1X_2+Y_1Y_2$.

But the integrating density functions distributed random variables $X_1^2+X_2^2+Y_1^2+Y_2^2$ and $X_1X_2+Y_1Y_2$ are the same. These results can be reduced to consideration of the integrating density functions by the change of random variable in the algebraic system.
Above example, the integrating density function about $\cos\theta$ is under investigation.

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