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# Semiconvergence Spaces

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## 1. Introduction.

It is well known [1] that the convergence of filters on a topological space essentially determines its topological structures. In order to generalize topological structures, Fischer and Kent among others, have proposed various kinds of structures of the convergences of filters. In this note, we introduce the concept of semiconvergence structures, and show that initial and final structures in the sense of Bourbaki[1] of semiconvergence structures exist.

Moreover we show that the category of convergence spaces introduced by Kent[4], the category of limit spaces introduced by Fischer[2] and the category of topological spaces are all bireflective in the category of semiconvergence spaces and continuous maps.

## 2. Semiconvergences spaces.

**Definition.** Let X be a set, let F(X) be the set of all filters on X and let P(F(X)) be the power set of F(X). Then $c: X \longrightarrow P(F(X))$ is a *semiconvergence structure* on X if c satisfies the following two statements:

$C_1$) For each $x \in X$, $\dot{x} = \{A \subset X \mid x \in A\} \in c(x)$.

$C_2$) If $\mathcal{F} \in c(x)$ and $\mathcal{F} \subset \mathcal{G}$, then $\mathcal{G} \in c(x)$.

In this case (X, c) is called the *semiconvergence space.* If $\mathcal{F} \in c(x)$, then x is called a *limit* of $\mathcal{F}$, and $\mathcal{F}$ is said to converge to x, and we write $\mathcal{F} \longrightarrow x$.

**Definition.** Let (X, c) and (Y, c′) are the semiconvergence spaces. A map $f: X \longrightarrow Y$ is called *continuous* if $\mathcal{F} \in c(x)$ implies $f(\mathcal{F}) \in c'(f(x))$.

**Proposition.** *For each semiconvergence space* (X, c), *the identity map* $1_X: (X, c) \longrightarrow (X, c)$ *is continuous and if* $f: (X, c) \longrightarrow (Y, c')$ *is continuous and* $g: (Y, c') \longrightarrow (Z, c'')$ *is continuous then* $gf: (X, c) \longrightarrow (Z, c'')$ *is also continuous.*

**Remark.** By the above proposition, the class of all semiconvergence spaces and continuous maps form a category which will be denoted by <u>SConv.</u>

**Definition.** For any $(X, c) \in$ <u>SConv</u> and $A \subset X$, the set $\bar{A} = \{x \in X \mid$ there is a filter $\mathcal{F}$ on X with $\mathcal{F} \in c(x)$ and $A \in \mathcal{F}\}$. is called the *closure* of A.

Remark. $\bar{A} = \{x \in X \mid$ there is an ultrafilter $\mathcal{U}$ on X with $\mathcal{U} \in c(x)$ and $A \in \mathcal{U}\}$.

The following proposition is then obvious by the above remark.

**Proposition.** *Let* (X, c) *belong to* <u>SConv</u>. *Then* 1) $\overline{\phi}=\phi$, 2) $A\subset\overline{A}$, 3) $\overline{A\cup B}=\overline{A}\cup\overline{B}$, 4) *In general,* $\overline{\overline{A}}\neq\overline{A}$.

**Theorem.** *Let* $((X_i, c_i))_{i\in I}$ *be a family in* <u>SConv</u> *and* $f_i : X\longrightarrow X_i$ *a map for each* $i\in I$. *Then there is a semiconvergence structure* c *on* X *such that* $(f_i : (X, c)\longrightarrow(X_i, c_i))_{i\in I}$ *is initial, in the sense of Bourbaki, i.e., for each* $i\in I$, $f_i$ *is continuous, and for any semi convergence space* (Y, c′) *a map* $g : (Y, c')\longrightarrow(X, c)$ *is continuous iff for each* $i\in I$, $f_ig : (Y, c')\longrightarrow(X_i, c_i)$ *is continuous.*

**Proof.**: Define c as follows: A filter $\mathcal{F}$ belongs to c(x) for some $x\in X$ iff for each $i\in I$, $f_i(\mathcal{F})$ belongs to $c_i(f_i(x))$. For each $x\in X$, $f_i(\dot{x})=\dot{\widehat{f_i(x)}}$; hence $\dot{x}\in c(x)$. If $\mathcal{F}\in c(x)$ and $\mathcal{F}\subset\mathcal{G}$, then $f_i(\mathcal{F})\in c_i(f_i(x))$ and $f_i(\mathcal{F})\subset f_i(\mathcal{G})$ and hence $\mathcal{G}\in c(x)$. Thus c is a semiconvergence structure on X. Obviously for each $i\in I$, $f_i$ is continuous.

Suppose for each $i\in I$, $f_ig : (Y, c')\longrightarrow(X_i, c_i)$ is continuous and suppose $\mathcal{F}\in c'(y)$. Then for each $i\in I$ $f_ig(\mathcal{F})=f_i(g(\mathcal{F}))\in c_i(f_i(g(y))$. Hence $g(\mathcal{F})\in c(g(y)))$. This completes the proof.

**Definition.** The semiconvergence structure defined in the above theorem is called the *initial semiconvergence structure with respect to* $(f_i)$.

**Definition.** For a semiconvergence space (X, c) and a subset A of X, the initial convergence structure $c_A$ with respect to the natural embedding $A\longrightarrow X$ is called the *relative semiconvergence structure of c on A*, and $(A, c_A)$ is called a *subspace* of (X, c).

**Remark.** A filter $\mathcal{F}$ on A belongs to $c_A(x)$ iff the filter generated by $\mathcal{F}$ on X belongs to c(x).

**Definition.** Let $((X_i, c_i))_{i\in I}$ be a family of semiconvergence spaces indexed by a set I. Then the initial semiconvergence structure $\prod c_i$ on $\prod X_i$ with respect to projections is called the *product semiconvergence structure* and $(\prod X_i, \prod c_i)$ is called the *product semiconvergence space of the family.*

**Remark.** A filter $\mathcal{F}$ on $\prod X_i$ belongs to $\prod c_i(x)$ iff for each projection $Pr_i$, $Pr_i(\mathcal{F})\in c_i(Pr_i(x))$.

The following proposition is immediate by the above theorem.

**Proposition.** *A map* $f : (X, c)\longrightarrow(\prod X_i, \prod c_i)$ *is continuous iff for each* $i\in I$, $Pr_if : (X, c)\longrightarrow(X_i, c_i)$ *is continuous. Hence* $(\prod X_i, \prod c_i)$ *is a categorical product of* $((X_i, c_i))_{i\in I}$ *in* <u>SConv.</u>

**Theorem.** *Let* $((X_i, c_i))_{i\in I}$ *be a family of semiconvergence spaces and for each* $i\in I$, $f_i : x_i\longrightarrow X$ *a map. Then there is a semiconvergence structure* c *such that* $(f_i : (X_i, c_i)\longrightarrow(X, c))_{i\in I}$ *is final in the sense of Bourbaki.*

**Proof.** Define c as follows: $\mathcal{F}\in c(x)$ iff either $\mathcal{F}=\dot{x}$ or there are $i\in I$ and $\mathcal{K}\in c_i(y)$ such that $f_i(\mathcal{K})\subseteq\mathcal{F}$ and $f_i(y)=x$. Then it is obvious that $\dot{x}\in c(x)$ for each $x\in X$.

Suppose $\mathcal{F}\in c(x)$ and $\mathcal{F}\subseteq\mathcal{G}$. If $\mathcal{F}=\dot{x}$, then $\mathcal{F}=\mathcal{G}=\dot{x}\in c(x)$. If there is $\mathcal{K}\in c_i(y)$ for some $i\in I$ such that $f_i(\mathcal{K})\subseteq\mathcal{F}$ and $f_i(y)=x$, then $f_i(\mathcal{K})\subseteq\mathcal{F}\subseteq\mathcal{G}$ and hence $\mathcal{G}\in c(x)$. Hence c is a semiconvergence structure on X. It is obvious that for each $i\in I$, $f_i : (X_i, c_i)\longrightarrow(X, c)$

is continuous. Suppose for a semiconvergence space $(Y, c')$ and a map $g: X \longrightarrow Y$, $gf_i: (X_i, c_i) \longrightarrow (Y, c')$ is continuous, and $\mathcal{F} \in c(x)$. If $\mathcal{F} = \dot{x}$, then obviously $g(\mathcal{F}) \in c'(g(x))$. If there is $\mathcal{K} \in c_i(y)$ for some $i \in I$ with $f_i(\mathcal{K}) \subseteq \mathcal{F}$ and $f_i(y) = x$. then $gf_i(\mathcal{K}) \in c'(f_i(g(y)))$ and hence $g(\mathcal{F}) \in c'(g(x))$. Thus g is also continuous.

**Definition.** The semiconvergence structure c defined in the above theorem is called the *final structure on X with respect to $(f_i)$*.

Using the final structures, one can define quotient spaces and coproducts in SConv.

**Definition.** 1) For a set X, a map $c: X \longrightarrow P(F(X))$ is called a *convergence structure on X* if c is a semiconvergence structure and for any $\mathcal{F} \in c(x)$, $\mathcal{F} \cap \dot{x} \in c(x)$. And $(X, c)$ is called a *convergence space.*

2) A semiconvergence structure on a set X is called a *limit structure on X* if $\mathcal{F}, \mathcal{G} \in c(x)$ implies $\mathcal{F} \cap \mathcal{G} \in c(x)$.

It is known [4] that the category Conv of convergence spaces and continuous maps contains the category Lim of limit spaces and the category Top of topological spaces as bireflective subcategories.

**Theorem.** *The category Conv is bireflective in SConv.*

**Proof.** It is enough [3] to show that Conv is closed under the formation of initial sources in SConv. Let $(f_i: (X, c) \longrightarrow (X_i, c_i))$ be an initial source in SConv such that for each $i \in I$, $(X_i, c_i) \in$ Conv. Suppose $\mathcal{F} \in c(x)$. Then for each $i \in I$, $f_i(\mathcal{F}) \in c(f_i(x))$ and hence $f_i(\mathcal{F}) \cap \widehat{f_i(x)}$ also belongs to $c(f_i(x))$.

Since $f_i(\mathcal{F} \cap \dot{x}) = f_i(\mathcal{F}) \cap \widehat{f_i(x)}$, $\mathcal{F} \cap \dot{x}$ also belongs to $c(x)$. Thus $(X, c)$ belongs to Conv.

**Remark.** 1) For any $(X, c) \in$ SConv, the Conv-reflection of $(X, c)$ is given as follows: define $c_r$ by $\mathcal{F} \in c_r(x)$ iff there is $\mathcal{G} \in c(x)$ with $\mathcal{G} \cap \dot{x} \subseteq \mathcal{F}$, and then the identity map $(X, c) \longrightarrow (X, c_r)$ is the desired reflection.

2) Lim and Top are also bireflective in SConv.

### References


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[3] H. Herrlich (1974), Topological Structures, *Math. Centre Tracks,* 52, 59-122.

[4] D. Kent (1964), Convergence functions and their related topology, *Fund. Math.* 54, 125-133.