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On A Semitopological Semigroup

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1. Introduction

In this paper I investigated some properties of translational hull of a semigroup.

If S is a semigroup, then a function λ:S$\longrightarrow$S is a *left translation* of S if, for all x, y$\in$S, λ(xy)=(λx)y and a function ρ : S$\longrightarrow$S is a *right translation* of S if, for all x, y$\in$S, (xy)ρ=x(yρ). A left translation λ and a right translation ρ are said to be *linked* if x(λy)=(xρ)y, for all x, y$\in$S and the linked pair (λ, ρ)is called a *bitranslation* of S. If $\omega=(\lambda,\rho)$ is a bitranslation of S and a$\in$S, then we denote ω a=λa and aω=aρ.

For x, y$\in$S we have ω(xy)=(ωx)y, (xy)ω=x(yω), and x(ωy)=(xω)y. Clearly Λ(S) and P(S) of all left and right translations of S, respectively, are semigroups with respect to composition of functions and Ω(S) is a subsemigroup of Λ(S)$\times$P(S). The semigroup Ω(S) is called the *translational hull* of S.

If a semigroup S is endowed with a topology and $\omega=(\lambda,\rho)$ is a bitranslation of S, ω is a *continuous bitranslation* if λ and ρ are both continuous. For each a$\in$S, we use $\lambda_a(\rho_a)$ to denote the left (right) translation x$\longrightarrow$ax(x$\longrightarrow$xa). Then λ_a and ρ_a are linked for each a$\in$S, and hence $\omega_a=(\lambda_a, \rho_a)$ is a bitranslation of S. For a$\in$S, the translations λ_a and ρ_a are called *inner left* and *inner right translations* respectively and ω_a is called an *inner bitranslation.*

The set π(S) of all inner bitranslation of S is a subsemigroup of Ω(S) and the function π : S$\longrightarrow\pi$(S) defined by $\pi(a)=\omega_a$ is a homomorphism. The semigroup π(S) is called the *inner translational hull* of S and the homomorphism π is called the *canonical homomorphism.* Let Λ_p(S) and P_p(S) are denoted by the semigroups Λ(S) and P(S), respectively, endowed with the relative topology of pointwise convergence on S^s and Ω_p(S) be the semigroup Ω(S) with the product topology on $\Lambda_p(S)\times P_p(S)$. Also, Ω_c(S) be the semigroup Ω(S) with the topology of continuous convergence. A semigroup S is said to *act* on a set X if there exists a function π:X$\times$S$\longrightarrow$X (X is a topological space) satisfying π(x, st)=$\pi(\pi$(x, s), t) for all s, t$\in$S, x$\in$X.

Let π(x, s)=xs. The function π is called a (left) *action* of S on X.

2. Translational hull

A semigroup on a Hausdorff space is called a *semitopological semigroup* if multiplication is separately continuous, and is called a *topological semigroup* if multipication is jointly continuous.

Lemma 2.1 ([2]) *Let* S *be a semitopological semigroup. Then the canonical homomorphism* $\pi: S \longrightarrow \Omega_p(S)$ *is continuous.*

Lemma 2.2 ([2]) *Let* S *be a semigroup on a topological space. Then the multipication of* $\Omega_c(S)$ *is continuous.*

A semigroup S is said to be *left* (*right*) *reductive* if xa=xb (ax=bx) for all $x \in S$ implies a=b, and is said to be *reductive* if S is both left and right reductive. A semitopological semigroup S is said to be *left* (*right*) *net reductive* if $xa_\alpha \longrightarrow xa$ ($a_\alpha x \longrightarrow ax$) for all $x \in S$ implies that $a_\alpha \longrightarrow a$, and S is said to be *net reductive* if S is both left and right net reductive. A semitopological semigroup S is saidto be *left* (*right*) *bi-net reductive* if for a net a_α in S and $a \in S$, the condition that $x_\beta a_\alpha \longrightarrow xa$ ($a_\alpha x_\beta \longrightarrow ax$) for $x_\beta \longrightarrow x$ in S implies $a_\alpha \longrightarrow a$.

And S is said to be *bi-net reductive* if S is both left and right bi-net reductive.

Lemma 2.3 ([2]) *In a semitopological semigroup* S, *net reductivity implies bi-net reductivity which in turn implies reductivity.*

Theorem 2.4 *Let* S *bw a semitopological semigroup. Then the canonical homomorphism* $\pi: S \longrightarrow \Omega_p(S)$ *is both an isomorphism and a homeomorphism if and only if* S *is net reductive.*

Proof. Suppose S be a semitopological semigroup which is net reductive. By Lemma 2.1, $\pi: S \longrightarrow \Omega_p(S)$ is a continuous homomorphism. From Lemma 2.3, S is bi-net reductive and hence is monomorphism. Suppose now that a_α is a net in S such that $xa_\alpha \longrightarrow xa$ for all $x \in S$ and $a \in S$. For constant net $x_\beta = x$, $x_\beta a_\alpha \longrightarrow xa$. By bi-net reductivity, a_α converges to a. It follows that $\pi^{-1}: \pi(S) \longrightarrow S$ is continuous ($\pi(S)$ with the reductive topology of $\Omega_p(S)$) and π is a homeomorphism into $\Omega_p(S)$.

Conversely suppose that $\pi: S \longrightarrow \Omega_p(S)$ is a homeomorphism. And suppose that a_α is a net in S, $a \in S$ such that $xa_\alpha \longrightarrow xa$ for all $x \in S$. By definition of the topology of $\Omega_p(S)$, $\lambda_\alpha(a_\alpha) \longrightarrow \lambda_a(a)$ in $\Omega_p(S)$. Since π is a homeomorphism, we have $a_\alpha \longrightarrow a$.

Similarly for the right.

Theorem 2.5 *Each bitranslation of a bi-net reductive semitopological semigroup* S *is continuous.*

Proof. Let ω be a bitranslation and let y_α be a net in S converging to y. Then for each $x_\beta \longrightarrow x$ in S, $x_\beta(\omega y_\alpha) \longrightarrow (x\omega)yx(\omega y)$. Since S is bi-net reductive and for $x_\beta \longrightarrow x$ in S, $\omega y_\alpha \longrightarrow \omega y$ and $y_\alpha \omega \longrightarrow y\omega$. Hence ω is continuous.

Theorem 2.6 *Let* S *be a compact net reductive semitopological semigroup. Then* $\Omega_c(S)$ *is a compact topological semigroup.*

Proof. $\Omega(S)$ is embedded in $\pi\{S \times S\}_{a \in S}$ by $\omega \longrightarrow (\omega a, a\omega)$ in the a-th coordinate for each $a \in S$. We have to show that $\Omega(S)$ embedded is a closed subset of $\pi\{S \times S\}_{a \in S}$. Let ω_α be a net in $\Omega(S)$ convergent to an element ω of $\{S \times S\}_{a \in S}$. Let ω be a bifunction by defining ωx to be the first term in the x-th coordinate of $\pi\{S \times S\}_{a \in S}$ and $x\omega$ the second. Let $x, y \in S$. Then $x(\omega y) = x(\lim \omega_\alpha y) = \lim x(\omega_\alpha y) = \lim (x\omega_\alpha)y = (\lim x\omega_\alpha)y = (x\omega)y$. Hence ω is a linked pair. Since S is net reductive, ω is a bitranslation of S. In view of Theorem 2.5, ω is continuous and thus $\Omega(S)$ is closed in $\pi\{S \times S\}_{a \in S}$. Therefore $\Omega_c(S)$ is a closed subset of a compact space and hence compact. By Lemma 2.2, multiplication on $\Omega_c(S)$ is continuous. Hence $\Omega_c(S)$ is a topological semigroup.

3. Continuity

A semigroup S on a topological space is a *left (right) semitopological semigroup* if the multiplication function is left (right) continuous. For $x, y \in X$, define $C(x, y) = \{s \in S : xs \neq ys\}$.

Lemma 3.1 ([4]) *Let* S *be a left semitopological semigroup,* X *a Hausdoroff space and* $\pi : X \times S \longrightarrow X$ *a left separately continuous action. If for* $s \in S$, $x, y \in X$, $y \neq xs$, *there exists* $r \in C(y, xs)$ *such that* π *is continuous at* (x, sr), *then there exists open sets* U, W, V *such that* $x \in U$, $s \in W$, $y \in V$ *and* $\pi(U \times W) \cap V = \phi$.

Theorem 3.2 *Let* S *be a left semitopological semigroup,* X *a compact Hausdorff space and* $\pi : X \times S \longrightarrow X$ *a left separately continuous action. Let* $(x, s) \in X \times S$. *If for each* $y \neq xs$, *there exists* $r \in C(y, xs)$ *such that* π *is continuous at* (x, s).

Proof. Let T be an open set containing xs. Then $X \backslash T$ is compact. By Lemma 3.1, each $y \in X \backslash T$, there exist open sets U_y, V_y, W_y such that $x \in U_y$ $s \in V_y$, $y \in W_y$ and $\pi(U_y \times V_y) \cap W_y = \phi$. A finite number of $\{W_y : y \in X \backslash T\}$ cover $X \backslash T$. Let U be the intersection of the corresponding U_y and V be the intersection of the corresponding V_y. Then U and V are open, $x \in U$, $s \in V$ and $\pi(U \times V) \subset T$.

Theorem 3.3 *Let* S *be a compact Hausdorff left semitopological semigroup with the identity* i, X *a compact Hausdoroff space, and* $\pi : X \times S \longrightarrow X$ *a separately continuous action. If* u *is a unit in* S, *then* π *is continuous at* (x, s) *for all* $x \in X$.

Proof. By hypothesis, there exists $u^{-1} \in S$ such that $u^{-1}u = uu^{-1} = i$. Since $ixs = ix(is) = i(xis) \in iX$, $\pi(iX \times S) \subset iX$, and $\pi|_{iX \times S}$ is continuous at (y, u) for all $y \in iX$. Since $ixsu^{-1}u = i(xs)i = xs$, define the composition $X \times S \rightarrow iX \times S \rightarrow X \rightarrow X$ by $(x, s) \rightarrow (ix, su^{-1}) \rightarrow ixsu^{-1} \rightarrow ixsu^{-1}u$.

Since the composition $(x, u) \rightarrow (ix, uu^{-1}) = (ix, i) \rightarrow (ix)i = ix \rightarrow ixu = xu$ is continuous, π is continuous at (x, u).

Theorem 3.4 *Let* S *be a locally compact Hausdorff left semitopological semigroup,* X *a compact metric space and* $\pi : X \times S \longrightarrow X$ *a separately continuous action. If there exists* $s \in S$ *such that* $aS = S$ *and* $\pi(S \times s) = X$, *then* π *is continuous at* (x, s) *for each* $x \in X$.

Proof. Let $x \in X$, and suppose $y \neq xs$. There exist open sets U and V such that $y \in U$, $xs \in V$ and $U \cap V = \phi$. By hypothesis, there exists $z \in X$ such that $y = zs$. Hence there exists open W, $s \in W$ such that $\pi(z \times W) \subset U$. And there exists $t \in W$ such that π is continuous at (x, t). By hypothesis, there exists $r \in S$ such that $t = sr$.

Then π is continuous at (x, sr). Thus $xsr = xt \in \pi(X \times W) \subset V$ and $yr = zsr = zt \in \pi(z \times W) \subset U$. Hence $yr \neq xsr$. By Theorem 3.2, π is continuous at (x, s).

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