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On The Regular Convergence Spaces

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Introduction

There have been many efforts made to introduce new approaches to topological problems, usually generalizing the notion of topology, proximity or uniformity. Among these, the concept of convergence structures has proved fruitful in recent years.

In particular, it is known [6] that the category of limit spaces and continuous maps is a cartesian closed topological category and it contains the category *Top* of topological spaces as a bireflective subcategory. Also, these have been many attempts to define concepts of separation axioms in convergence spaces.

In this paper, we define a concept of regular convergence space. Namely, a filter converges to a point if and only if the filter generated by closures of members of the filter also converges to the point. Since Herrlich has introduced the concept of topological functors ([4], [5]), it is known that the concept is just right one for the study of topology. Using his results, we show that the subcategory *RegConv* of regular convergence spaces is closed under initial sources in the category *Conv* of convergence spaces and continuous maps. Thus we conclude that the category *RegConv* is bireflective in *Conv* and is also a properly fibred topological category.

Finally, observing *RegConv* ∩ *Top* is precisely the category *Reg* regular topological spaces, *Reg* is also bireflective in *Top* and is a properly fibred topological category.

1. Convergence Structures and Topological Categories

In this section, we include basic properties of topological categories for the further development. We omit their proof and refer to [1] for filters [3], [7] for convergence structures, and [4], [5] for topological functors and [1] for filters.

1.1 Definition (a) Let X be a set and F(X) be the set of all filters on X. A map $c: X \longrightarrow P(F(X))$ is called a *convergence structure* on X if c satisfies the following:

1) For any $x \in X$, $\dot{x} \in c(x)$, where $\dot{x}$ denote the ultrafilter generated by $\{x\}$.

2) For any $\mathcal{F} \in c(x)$ and $\mathcal{F} \subseteq \mathcal{G}$, $\mathcal{G} \in c(x)$.

3) For any $\mathcal{F} \in c(x)$, $\mathcal{F} \cap \dot{x} \in c(x)$.

In this case, (X, c) is called a *convergence space* and X, an *underlying set* of the space (X, c), For $F \in c(x)$, x is called a *limit* of $\mathcal{F}$ and $\mathcal{F}$ is said to *converges* to x.

(b) Instead of [3] if c satifies:

(L) For any $\mathcal{F}, \mathcal{G} \in c(x)$, $\mathcal{F} \cap \mathcal{G} \in c(x)$, then c is called a *limt structure* on X and (X, c) is called a *limit space.*

1.2 Remark (a) Every limit space is a convergence space.

(b) For any topological space $(X, \mathcal{T})$, if we define c on X by $\mathcal{F} \in c(x)$ iff $\mathcal{F}$ converges to x in the space, then (X, c) is a limit space.

1.3 Definition Let (X, c) be a convergence space. For any $A \subset X$, the set $\bar{A} = \{x \in X \mid$ there is a filter $\mathcal{F} \in c(x)$ with $A \in \mathcal{F}\}$ is called the *closure* of A.

It is straight forward that for any convergence space (X, c), $\bar{\phi} = \phi$, $A \subseteq \bar{A}$ and $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

However in general $\bar{\bar{A}} \neq \bar{A}$.

1.4 Definition Let (X, c), (Y, c′) be convergence spaces. Then a map $f: X \longrightarrow Y$ is called *continuous* on (X, c) to (Y, c′) if for any $\mathcal{F} \in c(x)$ in X, $f(\mathcal{F}) \in c(f(x))$ in Y.

It is also obious that the class of all convergence spaces and continuous maps between them forms a category, which will be denoted by *Conv*. Then the category *Lim* of all limit spaces and continuous maps and the category *Top* of topological spaces and continuous maps are subcategories of *Conv*.

1.5 Definition Let A be a concrete category. Let X be a set, and (Y_i, η_i) an object for each $i \in I$ (I may be a proper class or set). Let $f_i : X \longrightarrow Y_i$ be a map for each $i \in I$. An A-structure ξ on X is *initial* with respect to

$(X, (f_i)_{i \in I}, (Y_i, \eta_i)_{i \in I})$ iff the following conditions hold:

(a) $f_i : (X, \xi) \longrightarrow (Y_i, \eta_i)$ is an A-morphism for each $i \in I$.

(b) If (Z, ζ) is an A-object, and $g: Z \longrightarrow X$ is a map such that for each $i \in I$, $f_i \circ g : (Z, \zeta) \longrightarrow (Y_i, \eta_i)$ is an A-morphism, then $g: (Z, \zeta) \longrightarrow (X, \xi)$ is an A-morphism.

In *Conv*, for a source $(f_i : (X, c) \longrightarrow (X_i, c_i))$ to be initial it is necessary and sufficient that $\mathcal{F} \in c(x)$ in X iff for each $i \in I$, $f_i(F) \in c_i(f_i(x))$.

1.6 Definition A concrete category A is called *topological* iff the following conditions are satisfied:

(a) A has initial structures. That is, for any set X, any family $(Y_i, \eta_i)_{i \in I}$ of A-objects and any family $(f_i : X \longrightarrow Y_i)_{i \in I}$ of maps, there exists an A-structure ξ on X which is initial with respect to $(X, (f_i)_{i \in I}, (Y_i, \eta_i)_{i \in I})$.

(b) A is properly fibred. That is (i) for every set X, the class of all A-structures on X (called the A-fibre of X) is a set and (ii) if ξ and η are A-structures on X so that $1_X : (X, \xi) \longrightarrow (X, \eta)$ and $1_X : (X, \eta) \longrightarrow (X, \xi)$ are A-morphisms, then $\xi = \eta$.

The category of *Conv* and *Top* are topological.

1.7 Theorem *Let* A *be a topological category. Then*

(a) A *is complete. Limits are constructed by giving the appropriate initial structure to th*

corresponding limit in Set.

(*b*) A *is cocomplete. Colimits are constructed by giving the appropriate final structure to the corresponding colimit in Set.*

1.8 Proposition *A properly fibred topological category is well-powered and cowell-powered.*

In the following, every subcategory will be assumed to be full isomorphism-closed.

1.9 Theorem *Let* B *be a subcategory of properly fibred topological category, Then the followings are equivalent:*

(*i*) B *is bireflective in* A.

(*ii*) B *is closed under initial source, i.e., for any initial source* $(X, f_i : X \longrightarrow X_i)_{i\in I}$ *with* $X_i \in B$ *for all* $i \in I$, X *belongs to* B.

(*iii*) B *is epireflective in* A *and contains all indiscrete objects of* A.

1.10 Theorem *If a subcategory* B *is bireflective in a properly fibred topological category* A, *then* B *is also a properly fibred topological category.*

The *Conv* is obiously a properly fibred topological category. Since *Lim* and *Top* are closed under initial sources in *Conv*, they are bireflective in *Top* and properly fibred topological categories.

2. Regular convergence spaces

2.1 Definition A convergence space (X, c) is said to be regular if ($\mathcal{F} \xrightarrow{c} x$ iff $\overline{\mathcal{F}} = \{\overline{F} \mid F \in \mathcal{F}\}$ $\overline{\mathcal{F}} \xrightarrow{c} x$)

2.2 Proposition *Let* (X, c) *be a convergence space in Top.* (X, c) *is regular if and only if* (X, c) *is regular in the above sense.*

Proof If (X, c) is topological regular, then for the nbd filter $\mathcal{N}(x)$, $\mathcal{N}(x) \longrightarrow x \Longrightarrow \{\overline{N} \mid \in \mathcal{N}(x)\} \longrightarrow x$. For all $V \in \mathcal{N}(x)$, there is $U \in \mathcal{N}(x)$ with $U \subset \overline{U} \subseteq V$. i.e. (X, c) is topologically regular.

For the converse, if $\mathcal{F} \longrightarrow x$, for any nbd N of x, there exist nbd V of x such that $V \subset \overline{V} \subseteq N$. And there is $F \in \mathcal{F}$ with $F \subset V$, so we have $\overline{F} \subset \overline{V} \subset N$ i.e. $N \in \overline{\mathcal{F}}$ and $\mathcal{F} \xrightarrow{-} x$.

2.3 Theorem *RegConv is bireflective in Conv.*

Proof Since *Conv* is properly fibred topological category, it is enough to show *RegConv* is closed under initial sources. Let $(f_i : (X, c) \longrightarrow (X_i, c_i))_{i\in I}$ be an initial source in *Conv* and for each $i \in I$, $(X_i, c_i) \in$ *RegConv*. Suppose $\mathcal{F} \longrightarrow x$ in (X, c). Then for each $i \in I$, $f_i(\mathcal{F}) \longrightarrow f(x)$ in (X_i, c_i); hence $\overline{f_i(\mathcal{F})} \longrightarrow f_i(x)$ in (X_i, c_i). For each $F \in \mathcal{F}$, $f_i(\overline{F}) \subseteq \overline{f_i(F)}$, $\overline{f_i(F)} \subseteq f_i(\overline{F})$. Thus $f_i(\overline{\mathcal{F}}) \longrightarrow f_i(x)$ in (X_i, c_i). Thus $\overline{\mathcal{F}} \longrightarrow x$ in (X, c).

2.4 Corollary *RegConv is productive and hereditary in Conv. and every indiscrete convergence space is regular.*

2.5 Corollary *Regconv is also properly fibred topological category.*

2.6 Corollary *RegConv* $\cap$ *Top* $\equiv$ *RegTop is also bireflective in Conv, hence in Top.*

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