A Note on Independence of Quadratic Forms

Sung-Shin Han*

This note is to demonstrate that the extention of Theorem 4. 15 of Graybill [1] on the independence of quadratic forms is possible. For the self-contained exposition, Theorem 4. 15 is rewritten.

Theorem 4.15 If $\boldsymbol{Y}$ is distributed $N(\mu, \sigma^2\boldsymbol{I})$, the positive semidefinite quadratic forms $\boldsymbol{Y'AY}$and $\boldsymbol{Y'BY}$ are independent if $\mathrm{tr}(\boldsymbol{AB})=0$.[1]

Theorem 4. 15 may be extended as follows:

Extention Theorem If $\boldsymbol{Y}$ is distributed $N(\mu, \sigma^2\boldsymbol{I})$, the positive semidefinite quadratic forms $\boldsymbol{Y'AY}$ and $\boldsymbol{Y'BY}$ are independent if and only if $\mathrm{tr}(\boldsymbol{AB})=0$.

It is easy to see that the following theorem together with Theorem 4. 10 of Graybill[2] proves Extention Theorem.

Theorem $\mathrm{tr}(\boldsymbol{AB})=0$ if and only if $\boldsymbol{AB}=\boldsymbol{0}$ where

$\boldsymbol{A}$ and $\boldsymbol{B}$ are positive semidefinite and symmetric.

<Proof>

($\Rightarrow$) Let $\boldsymbol{A}=\boldsymbol{P'}\Lambda_A\boldsymbol{P}$ and $\boldsymbol{B}=\boldsymbol{Q'}\Lambda_B\boldsymbol{Q}$ where Λ_A and Λ_B are diagonal matrices whose diagonal elements are eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Then,

$$
\begin{aligned}
(1)\quad \mathrm{tr}(\boldsymbol{AB}) &= \mathrm{tr}(\boldsymbol{P'}\Lambda_A\boldsymbol{PQ'}\Lambda_B\boldsymbol{Q}) \\
&= \mathrm{tr}(\Lambda_A\boldsymbol{PQ'}\Lambda_B\boldsymbol{QP'}) \\
&= \mathrm{tr}(\Lambda_A\boldsymbol{R'}\Lambda_B\boldsymbol{R}) \quad \text{where } \boldsymbol{R}=\boldsymbol{QP'}
\end{aligned}
$$

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* Assistant Professor of Economics, Yonsei University

1) $\boldsymbol{Y}$ denotes an $n\times 1$ random vector. $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n\times n$ positive semidefinite and symmetric matrices. $N(\boldsymbol{\mu}, \sigma^2\boldsymbol{I})$ denote the multivariate normal distribution with mean μ and variance-covariance matrix $\sigma^2\boldsymbol{I}$. The $\mathrm{tr}(\cdot)$ denote trace which is defined as the sum of diagonal elements of a matrix.

2) Graybill, p.84
Theorem 4.10 : If $\boldsymbol{Y}$ is distributed $N(\boldsymbol{\mu}, \sigma^2\boldsymbol{I})$, the two positive semidefinite quadratic forms $\boldsymbol{Y'AY}$ and $\boldsymbol{Y'BY}$ are independent if and only if $\boldsymbol{AB}=\boldsymbol{0}$.

Rewериting (1),

$$(2)\ \ \mathrm{tr}(\boldsymbol{AB})=\mathrm{tr}(\Lambda_A\sum_{i=1}^{n}\lambda_{Bi}\boldsymbol{R}_{i\cdot}\boldsymbol{R}_{i\cdot}')$$

where λ_{Bi} and $\boldsymbol{R}_{i\cdot}$ are i-th diagonal element of Λ_B and the i-th column vector of $\boldsymbol{R}$, rerpectively.

Or,

$$(3)\ \ \mathrm{tr}(\boldsymbol{AB})=\sum_{i=1}^{n}\lambda_{Bi}\mathrm{tr}(\Lambda_A\boldsymbol{R}_{i\cdot}\boldsymbol{R}_{i\cdot}')$$

$$=\sum_{i=1}^{n}\lambda_{Bi}\boldsymbol{R}_{i\cdot}'\Lambda_A\boldsymbol{R}_{i\cdot}$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{Bi}\lambda_{Aj}(R_{ij})^2$$

where λ_{Aj} and R_{ji} are the j-th diagonal element of Λ_A and the j-th element of $\boldsymbol{R}_{i\cdot}$

Thus, by assumption,

$$(4)\ \ \sum_{i,j}\lambda_{Bi}\lambda_{Aj}(R_{ij})^2=0.$$

The positive semidefiniteness of $\boldsymbol{A}$ and $\boldsymbol{B}$ implies

$$(5)\ \ \lambda_{Bi}\geq 0,\ \ \lambda_{Aj}\geq 0;\ \ i,j=1,\ldots,n.$$

From (4) and (5),

$$(6)\ \ R_{ij}=0 \text{ if } \lambda_{Bi}\neq 0 \text{ and } \lambda_{Aj}\neq 0.$$

(4), (5), and (6) imply

$$(7)\ \ \lambda_{Bi}\lambda_{Aj}R_{ij}=0.$$

Note that $\lambda_{Bi}\lambda_{Aj}R_{ij}$ is the (j,i)-th element of $\Lambda_A\boldsymbol{R}'\Lambda_B$, which implies $\Lambda_A\boldsymbol{R}'\Lambda_B=0$. Thus, the conclusion is immediate.

($\Leftarrow$) It is self-obvious from the definition of trace.

REFERENCE

[1] Graybill, F.A., *An Introduction to Linear Statisitical Models,* Vol. 1, New York: McGraw-Hill, 1961.