Stochastic Square Duels With Continuous Interfiring Times

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I. INTRODUCTION

Many attempts have been made to analyze the reality of combat operations. In 1916 Lanchester [5] set up differential equations that can depict the macroscopic features of a combat where two opposing forces have large numbers of combatants.

In recent years, the theory of stochastic duels [1], [6] has been developed to describe encounters on a more microscopic scale. Weapon effectiveness parameters such as hit(kill) probability, conditional kill probability on hit, average rate of fire, cover, concealment, mobility, and so forth can be incorporated into the stochastic duel models and therefore these models may be useful in designing optimal levels of effectiveness parameters and evaluating firing tactics and strategies.

Ancker and Williams [2] constructed triangular (two versus one) or square (two versus two) duels where both duelists of one side fire simultaneously at fixed time intervals. They compared two square duels, one with both duelists of one side concentrating on an opponent and the other with each duelist of one side assigned to a seperate opponent, and concluded that the second case is superior to the first one as a firing strategy.

This paper presents general solutions for square duels listed in Table 1 with

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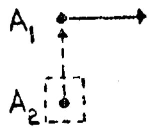
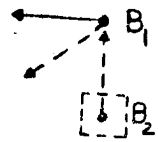
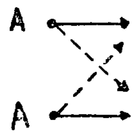
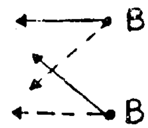
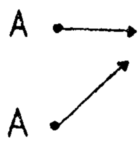
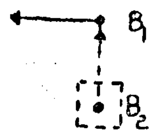
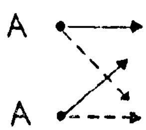
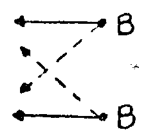
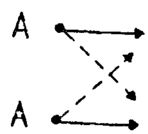
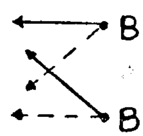
Table 1. **Square Duels with Various Strategy Combinations**

Type	Forces: Blue	Forces: Red	Firing strategy (t_B, t_R)	Duelists
1. Square duel-1			(S,S)	heterogeneous (A_1, A_2, B_1, B_2)
2. Square duel-2			(C,C)	homogeneous
3. Square duel-3			(C,S)	heterogeneous (B_1, B_2) homogeneous (A)
4. Square duel-4			(I,I)	homogeneous
5. Square duel-5			(I,C)	homogeneous
6. Square duel-6	random	random	(R,R)	homogeneous

continuous interfiring times and firing strategies such as standby (S), concentrated (C), individually seperated (I) and random (R) firings.

In each square duel, the winning probabilities of a given side with limited and unlimited duel times are obtained. Examples with negative exponentially distributed interfiring times are given and relative advantages among firing strategies are discussed.

II. PRELIMINARIES

A. Assumptions

We assume that:

(i) Both sides, Blue and Red, consist of two contestants each, and any side that kills both opponents before it loses its members will be declared the winner of the square duel.

(ii) Both sides have certain engagement strategies (t_B, t_R) where t_B and t_R indicate the firing strategies $(S), (C), (I)$ and (R) employed by Blue and Red respectively when they start the duel.

(iii) Each square duel has four different intermediate duel pairs denoted by (i_{t_B}, j_{t_R}) where Blue with i members and strategy t_B is counter-paired with Red with j members and strategy t_R. In each intermediate duel pair (i_{t_B}, j_{t_R}) both sides start the duel with unloaded weapons and unlimited ammunition supply and both duelists of each side are to fire simultaneously when strategy (C) or (I) is employed.

(iv) In case of heterogeneous duelists, both duelists A_1 and A_2 of the Blue have the fixed kill probabilities p_{A_1} and p_{A_2} and interfiring time probability density functions (pdf) $f(t;A_1)$ and $f(t;A_2)$ respectively, whereas we have $A_1 = A_2 = A$ in case of homogeneous duelists. p_{B_1}, p_{B_2}, p_B, $f(t;B_1)$, $f(t;B_2)$ and $f(t;B)$ are similarly defined for Red.

For standby strategy (S):

(v-S) A standby duelist participates in the duel only after the one previously engaged is killed and therefore only one versus one duel, i.e., "fundamental" duel is maintained until either side loses both members.

(vi-S) No time-delay is assumed in replacing the killed with the standby.

For concentrated strategy (C):

(v-C) Both duelists of one side concentrate their simultaneous firings on an opponent until it is destroyed.

(vi-C) Both duelists shift to the remaining target immediately after their aimed target is killed.

For individually seperated strategy (I):

(v-I) Each duelist of one side is seperately assigned to a different opponent.

(vi-I) If one of the opponents is killed, both duelists immediately concentrate their firings on the remaining opponent.

For random firing strategy(R):

(v-R) Both duelists of one side are simultaneously engaged in a square duel. However their firing strategies are randomly selected between strategy (C) and (I).

(vi-R) The kill processes immediately after an opponent is killed follow assumptions (vi-C) or (vi-I).

B. Nomenclature

We define that:

$P(T;\overline{t_B},t_R)$: The probability that Blue wins when the duel time is limited to T and both are engaged in a square duel (t_B,t_R). Here, the bar(—) indicates the winning side.

$P(T;t_B,t_R)$: The probability of a draw. We note that there is no superscript bar.

$P^*(s;\overline{t_B},t_R)$: The Laplace transform of $P(T;\overline{t_B},t_R)$.

$P(\overline{t_B},t_R)$: The probability that Blue wins when the duel time is unlimited.

$p(t;\overline{t_B},t_R)dt$: The probability that Blue wins in time interval $[t, t+dt]$ when both are engaged in a square duel (t_B,t_R).

$p(t;\overline{t_B},t_R,k)dt$: The probability that Blue with k losses wins in time interval $[t, t+dt]$ when both are engaged in a square duel (t_B,t_R).

$h(t;\overline{i_{t_B}},j_{t_R})$ The probability that Blue kills Red opponent(s) first during time interval $[t, t+dt]$ in an intermediate duel pair (i_{t_B}, j_{t_R}).

$h^*(s;\overline{i_{t_B}}, j_{t_R})$: The Laplace transform of $h(t;\overline{i_{t_B}}, j_{t_R})$.

$H(\overline{i_{t_B}}, j_{t_R})$: The probability that Blue kills Red opponent(s) first in a duel pair (i_{t_B}, j_{t_R}) with unlimited duel time.

$h(t;i_{t_B})dt$: The probability that Blue with i members and strategy t_B kills counter-paired passive target(s) in $[t, t+dt]$.

$h^*(s;i_{t_B})$: The Laplace transform of $h(t;i_{t_B})$.

$f^*(s;A_1)$: The Laplace transform of $f(t;A_1)$.

Similar quantities can be defined for Red side.

Now, we derive $h^*(s;i_{t_B}, j_{t_R})$ and $H(i_{t_B}, j_{t_R})$ as follows: By definitions, we have

(1) $h(t;\overline{i_{t_B}}, j_{t_R})\,dt = h(t;i_{t_B})\int_t^\infty h(\tau;j_{t_R})\,d\tau\ dt$, and

(2) $h^*(s;\overline{i_{t_B}}, j_{t_R}) = \int_0^\infty e^{-st}\ h(t;i_{t_B})\int_t^\infty h(\tau;j_{t_R})\,d\tau\ dt,$

$$= \int_0^\infty h(\tau;j_{t_R})\left[\int_0^\tau e^{-st}\ h(t;i_{t_B})\,dt\right] d\tau.$$

By the Mellin inversion integral [3],

$$\int_0^\tau e^{-st}\ h(t;i_{t_B})\,dt = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{h^*(s+z;i_{t_B})}{z}\ e^{z\tau}dz$$

where real line $z=c$ is chosen in such a way that all singularities lie on the left of line $z=c$.

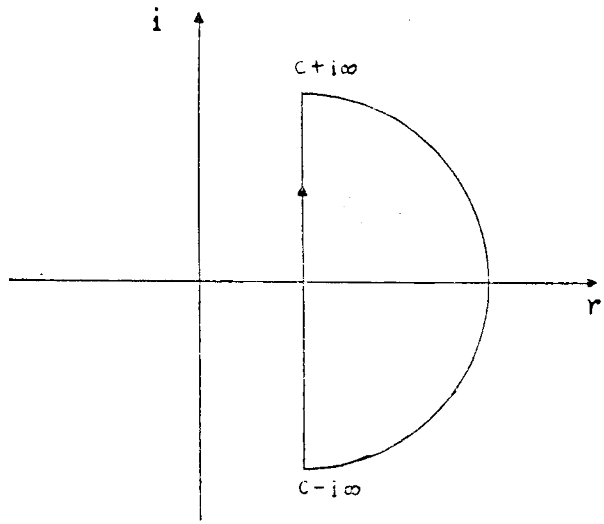


Fig. 1. Evaluation of Integral

Hence,

(3) $h^*(s;\overline{i_{t_B}}, j_{t_R}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h^*(s+z;i_{t_B})\left[\int_0^\infty h(\tau;j_{t_R})e^{z\tau}d\tau\right]\frac{dz}{z}$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h^*(s+z;i_{t_B})h^*(-z;j_{t_R})\frac{dz}{z}.$$

If we define $\lim_{s\to 0} h^*(s;i,j) = H(i_{t_B}, j_{t_R})$

(4) $H(\overline{i_{t_B}}, j_{t_R}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h^*(z;i_{t_B})h^*(-z;j_{t_R})\frac{dz}{z}$

$$= \int_0^\infty h(t;i_{t_B})\int_t^\infty h(\tau;j_{t_R})\,d\tau\ dt$$

With the above preliminaries, we will formulate various square duel models listed in Table 1.

III. SQUARE DUEL-1: (S,S)

In the "fundamental" duel of Williams and Ancker [6], the probability $h(t;A_i)dt$ that duelist A_i kills a passive target during time interval $[t, t+dt]$ and its Laplace transform $h^*(s;A_i)$ can be expressed as

$$(5)\quad h(t;A_i)\ dt=\sum_{n=1}^{\infty} p_{A_i}q_{A_i}^{n-1}\ f^{(n)}(t;A_i) \text{ and } h^*(s;A_i)=\frac{p_{A_i}\cdot f^*(s;A_i)}{1-q_{A_i}\cdot f^*(s;A_i)}$$

respectively where $p_{A_i}+q_{A_i}=1$ and $f^{(n)}(t;A_i)$ is the n-fold convolution of $f(t;A_i)$. Similar expressions can be obtained for B_j.

Then, with assumptions (i)-(iv), (v-S) and (vi-S), a standby square duel model with heterogeneous duelists can be formulated as follows.

$$(6)\quad P(T;\overline{S},S)=\int_0^T[p(t;\overline{S},S,0)+p(t;\overline{S},S,1)]\ dt$$

where

$$p(t;\overline{S},S,0)=(h(\overline{A_1},B_1)*h(\overline{A_1},B_2))(t),$$

$$p(t;\overline{S},S,1)=(h(\overline{A_1},B_1)*h(A_1,\overline{B_2})*h(\overline{A_2},B_2))(t)+ (h(A_1,\overline{B_1})*h(\overline{A_2},B_1)*h(\overline{A_2},B_2))(t)$$

and $*$ indicates a convolution.

Taking Laplace transform on both sides of equation (6), we have

$$(7)\quad P^*(s;\overline{S},S)=\frac{1}{s}\{h^*(s;\overline{A_1},B_1)\cdot h^*(s;\overline{A_1},B_2)+h^*(s;\overline{A_1},B_1)\cdot h^*(s;A_1,\overline{B_2})\cdot h^*(s;\overline{A_2},B_2)+h^*(s;A_1,\overline{B_1})\cdot h^*(s;\overline{A_2},B_1)\cdot h^*(s;\overline{A_2},B_2)\}$$

where $h^*(s;\overline{i}_{t_B},j_{t_R})$ is given by equation (3).

The inversion of equation (7) is generally not an easy task. Accordingly one may use numerical inversion techniques of Dubner and Abate [4] to compute $P(T;\overline{S},S)$ from $P^*(s;\overline{S},S)$.

From the final value theorem in the Laplace transform theory, that is,

(8) $\lim_{s \to 0} s \cdot \xi^*(s) = \lim_{T \to \infty} \xi(T)$,

the probability $P(\overline{S},S)$ can be obtained as

$$(9)\quad P(\overline{S},S) = \lim_{s \to 0} s \cdot P^*(s;\overline{S},S)$$
$$= H(\overline{A}_1,B_1) \cdot H(\overline{A_1},B_2) + H(\overline{A}_1,B_1) \cdot H(A_1,\overline{B}_2) \cdot H(\overline{A}_2,B_2) + H(A_1,\overline{B}_1) \cdot H(\overline{A}_2,B_1) \cdot H(\overline{A}_2,B_2)$$

where $H(\overline{i}_{t_B},j_{t_R})$ is given by equation (4).

The probability $P(T;S,S)$ of a draw is given by

$$(10)\quad P(T;S,S) = \int_T^\infty \{p(t;\overline{S},S,0) + p(t;\overline{S},S,1) + p(t;S,\overline{S},0) + p(t;S,\overline{S},1)\}\, dt.$$

Example-1

Let $f(t;A_i) = r_{A_i} e^{-r_{A_i} t}$ and $f(t;B_j) = r_{B_j} e^{-r_{B_j} t}$ for $i,j = 1, 2$ respectively. We note that r_{A_i} and r_{B_j} are the average rates of fire for A_i and B_j.

If we take Laplace transform on these and substitute them into equation (5) we have

$$(11)\quad h^*(s;A_i) = \frac{r_{A_i} p_{A_i}}{s + r_{A_i} p_{A_i}} \quad \text{and} \quad h^*(s;B_j) = \frac{r_{B_j} p_{B_j}}{s + r_{A_j} p_{A_j}}.$$

From equations (3), (7) and (11), we obtain

$$(12)\quad P^*(s;\overline{S},S) = \frac{1}{s}\left\{\left(\frac{u}{s+u+x}\right)\left[\left(\frac{u}{s+u+y}\right) + \left(\frac{y}{s+u+y}\right)\left(\frac{v}{s+v+y}\right)\right] + \left(\frac{x}{s+u+x}\right)\left(\frac{v}{s+v+x}\right)\left(\frac{v}{x+v+y}\right)\right\}$$

where $u = r_{A_1} p_{A_1}$, $v = r_{A_2} p_{A_2}$, $x = r_{B_1} p_{B_1}$ and $y = r_{B_2} p_{B_2}$.

From Heaviside expansion theorem [3], we have

$$\mathcal{L}^{-1}\left\{\prod_{i=0}^{n}\left(\frac{1}{s - r_i}\right)\right\} = \sum_{i=0}^{n}\prod_{j \neq i}^{n}\left(\frac{e^{r_i t}}{r_i - r_j}\right)$$

and

$$(13)\quad \mathcal{L}^{-1}\left\{\left(\frac{1}{s}\prod_{i=1}^{n}\left(\frac{1}{s - r_i}\right)\right)\right\} = \sum_{i=1}^{n}\prod_{j \neq i}^{n}\left\{\frac{e^{r_i t}}{r_i(r_i - r_j)}\right)\right\} + (-1)^n \prod_{i=1}^{n}\left(\frac{1}{r_i}\right)$$

where $\mathcal{L}$ indicates Laplace transform.

Then, $P^*(s;\overline{S},S)$ is inverted to yield

(14) $P(T;\overline{S},S)=C_0+C_1e^{-(u+x)T}+C_2e^{-(u+y)T}+C_3e^{-(v+x)T}+C_4e^{-(v+y)T}$

where

$$C_0=(\frac{u}{u+x})(\frac{u}{u+y})+(\frac{u}{u+x})(\frac{y}{u+y})(\frac{v}{v+y})+(\frac{x}{u+x})(\frac{v}{v+x})(\frac{v}{v+y})$$

$$C_1=\frac{u(u-v)[u(u+x-v-y)-vy]-v^2x(x-y)}{(u+x)(u-v)(u+x-v-y)(x-y)},$$

$$C_2=\frac{u^2(u-v)-uvy}{(u+y)(u-v)(y-x)},$$

$$C_3=\frac{v^2x}{(v+x)(u-v)(x-y)},$$

$$C_4=\frac{v[uy(x-y)+vx(u-v)]}{(v+y)(u+x-v-y)(u-v)(y-x)}.$$

Hence,

(15) $P(\overline{S},S)=C_0$

which can also be obtained from equations (8) and (12).

For the case of homogeneous standby duelists, i.e., $r_{A_1}p_{A_1}=r_{A_2}p_{A_2}=u$ and $r_{B_1}p_{B_1}=r_{B_2}p_{B_2}=x$, equation (15) reduces to

$$P(\overline{S},S)=(\frac{u}{u+x})^2+2(\frac{x}{u+x})(\frac{u}{u+x})^2$$

IV. SQUARE DUEL-2: (C,C)

We first define $h(t;2A_c)$ dt as the probability that both duelists of Blue with strategy (C) kills a passive opponent in time interval $[t, t+dt]$. Then, $h(t;2A_c)$ and its Laplace transform are

$$h(t;2A_c)dt=\sum_{n=1}^{\infty}(1-q_A^2)q_A^2f^{(n)}(t;A) \quad \text{and}$$

(16)
$$h^*(s;2A_c)=\frac{(1-q_A^2)f^*((s;A)}{1-q_A^2f^*(s;A)}$$

respectively. For Red side $h(t;2B_c)$ and $h^*(s;2B_c)$ are similarly defined.

With assumptions (i)-(iv) and (v-C)-(vi-C), we find

(17) $P(T;\bar{C},C)=\int_0^T\{p(t;\bar{C},C,0)+p(t;\bar{C},C,1)\}\,dt$

where

$$p(t;\bar{C},C,0)=(h(2\bar{A}_c,2B_c)*h(2\bar{A}_c,B))(t)$$

and

$$p(t;\bar{C},C,1)=(h(2\bar{A}_c,2B_c)*h(2A_c,\bar{B})*h(\bar{A},B))(t)+(h(2A_c,2\bar{B}_c)*h(\bar{A},2B_c)*h(\bar{A},B))(t).$$

Hence,

(18) $$P^*(s,\bar{C},C)=\frac{1}{s}\{h^*(s;2\bar{A}_c,2B_c)\cdot[(h^*(s;2\bar{A}_c,B)+h^*(s;2A_c,\bar{B})\cdot h^*(s;\bar{A},B)]+h^*(s;2A_c,2\bar{B}_c)\cdot h^*(s;\bar{A},2B_c)\cdot h^*(s;\bar{A},B)\},$$

and then, from equation (8), we have

(19) $$P(\bar{C},C)=H(2\bar{A}_c,2B_c)\cdot\{(H(2\bar{A}_c,B)+H(2A_c,\bar{B})\cdot H(\bar{A},B)\}+H(2A_c,2\bar{B}_c)\cdot H(\bar{A},2B_c)\cdot H(\bar{A},B).$$

Example-2

As in Example-1, interfiring times are assumed to be negative exponential. Then, from equation (16), we have

(20) $$h^*(s;2A_c)=\frac{r_A(1-q_A^2)}{s+r_A(1-q_A^2)}\text{ and }h^*(s;2B_c)=\frac{r_B(1-q_B^2)}{s+r_B(1-q_B^2)}.$$

Substituting equations (11) and (20) into equations (18)-(19), we can obtain $P(T;\bar{C},C)$ expressed as equation (14) with u,v,x and y replaced by $r_A(1-q_A^2)$, r_Ap_A, $r_B(1-q_B^2)$ and r_Bp_B respectively, and then $P(\bar{C},C)$ can be rewritten as

(21) $$P(\bar{C},C)=\left(\frac{r_A(1-q_A^2)}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)\left[\left(\frac{r_A(1-q_A^2)}{r_A(1-q_A^2)+r_Bp_B}\right)+\left(\frac{r_Bp_B}{r_A(1-q_A^2)+r_Bp_B}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_Bp_B}\right)\right]+\left(\frac{r_B(1-q_B^2)}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_B(1-q_B^2)}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_Bp_B}\right).$$

V. SQUARE DUEL-3: (C,S)

By following similar procedures as in section Ⅲ and Ⅳ, the winniing probability of Blue side with strategy (C) against Red with strategy (S) can be obtained as follows:

$$(22)\quad P(T;\bar{C},S)=\int_0^T \{p(\text{t};\bar{C},S,0)+P(t;\bar{C},S,1)\}\,dt$$

where $\quad p(t;\bar{C},S,0)=(h(2\bar{A}_c,B_1) * h(2\bar{A}_c,B_2))(t),$

and $\quad p(t;\bar{C},S,1)=(h(2\bar{A}_c,B_1) * h(2A_c,\bar{B}_2) * h(\bar{A},B_2))(t)+$

$$(h(2A_c,\bar{B}_1) * h(\bar{A},B_1) \ * h(\bar{A},B_2))(t).$$

Hence,

$$(23)\quad P^*(s;\bar{C},S)=\frac{1}{s}\ \{h^*(s;2\bar{A}_c,B_1)\cdot h^*(s;2\bar{A}_c,B_2)+h^*(s;2\bar{A}_c,B_1).\ h^*(s;2A_c,\bar{B}_2)$$
$$\cdot h^*(s;\bar{A},B_2)+h^*(s;2A_c,\bar{B}_1).\ h^*(s;\bar{A},B_1)\cdot h^*(s;\bar{A},B_2)\}$$

and

$$(24)\quad P(\bar{C},S)=H(2\bar{A}_c,B_1)\ \cdot\{H(2\bar{A}_c,B_2)+H(2A_c,\bar{B}_2)\cdot H(\bar{A},B_2)\}+$$
$$H\ (2A_c,\bar{B}_1)\cdot H(\bar{A},B_1)\cdot H(\bar{A},B_2).$$

For Blue side, we have

$$(25)\quad P^*(s;C,\bar{S})=(\frac{1}{s})\ \{h^*(s;2A_c,\bar{B}_1)\cdot h^*(s;A,\bar{B}_1)+h^*(s;2A_c,\bar{B}_1)$$
$$\cdot h^*(s;\bar{A},B_1)\cdot h^*(s;A,\bar{B}_2)+h^*(s;2\bar{A}_c,B_1)$$
$$\cdot h^*(s;2A_c,\bar{B}_2)\ \cdot h^*(s;A,\bar{B}_2)\},$$

and

$$(26)\quad P(C,\bar{S})=\ H(2A_c,\bar{B}_1)\{H(A,\bar{B}_1)+H(\bar{A},B_1)\cdot H(A,\bar{B}_2)\}$$
$$+H(2\bar{A}_c,B_1)\cdot H(2A_c,\bar{B}_2)\cdot H(A,\bar{B}_2).$$

Example-3

Let interfiring times be negative exponential as in Examples 1 and 2.

Putting equations (11) and (20) into equations (23)-(26), $P^*(s;\bar{C},S)$ and

$P(T;\bar{C},S)$ can be expressed as equations (12) and (14) where $r_A(1-q_A^2)$, r_Ap_A, $r_{B_1}p_{B_1}$, and $r_{B_2}p_{B_2}$ are substituted for u,v,x and y respectively, and then $P(\bar{C},S)$ can be rewritten as

$$(27)\quad P(\bar{C},S)=\left(\frac{r_A(1-q_A^2)}{r_A(1-q_A^2)+r_{B_1}p_{B_1}}\right)\left\{\left(\frac{r_A(1-q_A^2)}{r_A(1-q_A^2)+r_{B_2}p_{B_2}}\right)+\left(\frac{r_{B_2}p_{B_2}}{r_A(1-q_A^2)+r_{B_2}p_{B_2}}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_{B_2}p_{B_2}}\right)\right\}+\left(\frac{r_{B_1}p_{B_1}}{r_A(1-q_A^2)+r_{B_1}p_{B_1}}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_{B_1}p_{B_1}}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_{B_2}p_{B_2}}\right).$$

Similarly, for Red side, we obtain

$$(28)\quad P(C,\bar{S})=\left(\frac{r_{B_1}p_{B_1}}{r_A(1-q_A^2)+r_{B_1}p_{B_1}}\right)\left\{\left(\frac{r_{B_1}p_{B_1}}{r_Ap_A+r_{B_1}p_{B_1}}\right)+\left(\frac{r_Ap_A}{r_Ap_A+r_{B_1}p_{B_1}}\right)\left(\frac{r_{B_2}p_{B_2}}{r_Ap_A+r_{B_2}p_{B_2}}\right)\right\}+\left(\frac{r_A(1-q_A^2)}{r_A(1-q_A^2)+r_{B_1}p_{B_1}}\right)\left(\frac{r_{B_2}p_{B_2}}{r_A(1-q_A^2)+r_{B_2}p_{B_2}}\right)\left(\frac{r_{B_2}p_{B_2}}{r_Ap_A+r_{B_2}p_{B_2}}\right)$$

and it can be easily shown that $P(\bar{C},S)+P(C,\bar{S})=1$.

In particular, if Red's duelists are homogeneous and $r_A=r_B$, equations (27) and (28) reduce to

$$P(\bar{C},S)=\left(\frac{p_A}{p_A+p_B{}^2}\right)+\frac{p_A{}^2p_B(2-p_A)\{(2-p_A)(2p_A+p_B)+p_B\}}{\{p_A(2-p_A)+p_B\}^2\ (p_A+p_B)^2}$$

and

$$P(C,\bar{S})=2\left(\frac{p_B}{p_A(2-p_A)+p_B}\right)^2+\frac{p_B{}^2\{p_A{}^2(4-3p_A)+2p_Ap_B(1-p_A)-p_B{}^2\}}{\{p_A(2-p_A)+p_B\}^2\ (p_A+p_B)^2}$$

Then, the difference of the above two probabilities, i.e., the relative advantage of strategy (C) against strategy (S), is given by

$$(29)\quad D(C,S)\equiv P(\bar{C},S)-P(C,\bar{S})$$
$$[(p_A+p_B)\{(p_A(2-p_A)^2)(p_A+p_B)-p_B{}^2(p_A(2-p_A)+p_B)\}+(p_A-p_B)p_Ap_B\{(2-p_A)(2p_A+p_B)+p_B\}]/[\{p_A(2-p_A)+p_B\}^2\cdot(p_A+p_B)^2]$$

The above equations $P(\bar{C},S)$, $P(C,\bar{S})$ and $D(C,S)$ when $p_A=p_B=p$ are depicted in Fig. 2, which reveals that $D(C,S)\to\frac{5}{18}$ as $p\to 0$ and $D(C,S)\to 0$ as $p\to 1$, i.e., concentrated strategy is always more advantageous than standby strategy and the advantage becomes more pronounced for lower values of kill (hit) probabilities.

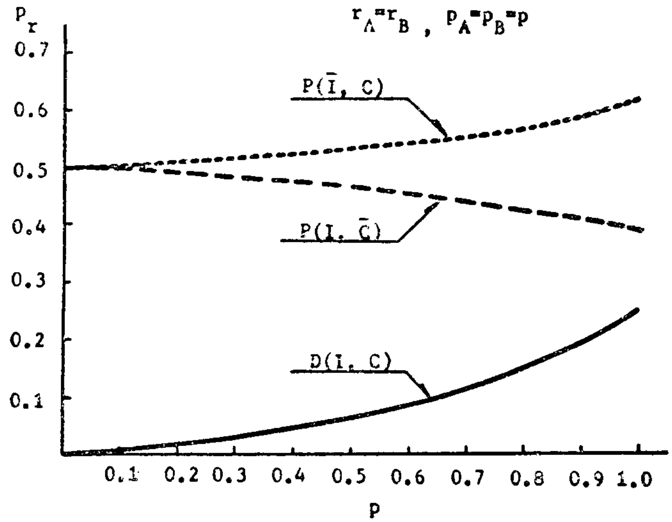


Fig. 2. Comparison of Concentrated Strategy against Standby Strategy

VI. SQUARE DUEL-4: (I,I)

This duel is similar to the second square duel of Ancker and Williams [2], except that it employs continuous interfiring times whereas their duel assumes fixed time intervals.

Let kill density functions $h(t;2A_I(2))$, $h(t;2A_I(1))$and $h(t;2A_I)$ be defined as follows:

$h(t;2A_I(2))\ dt$: The probability that both duelists of blue with strategy (I) kill both opponents in $[t,t+dt]$.

$h(t;2A_I(1))dt$: The probability that one of the Blue duelists kills one of the Red duelists in $[t,\ t+dt]$.

$h(t;2A_I)dt$: The probability that Blue with strategy (I) kills at least one of the opponents in $[t,\ t+dt]$.

Then, $h(t;2A_I(2))$ and $h(t;2A_I(1))$ and their Laplace transforms are given by

(30)
$$h(t;2A_I(2))=\sum_{n=1}^{\infty} p_A{}^2(q_A{}^2)^{n-1}f^{(n)}(t;A),$$
$$h(t;2A_I(1))=\sum_{n=1}^{\infty} 2\ p_Aq_A(q_A{}^2)^{n-1}f^{(n)}(t;A),$$

(31) $h^*(s;2A_I(2))=\dfrac{p_A{}^2f^*(s;A)}{1-q_A{}^2f^*(s;A)}$ and $h^*(s;2A_I(1))=\dfrac{2p_Aq_Af^*(s;A)}{1-q_A{}^2f^*(s;A)}$

where

$h(t;2A_I)=h(t;2A_I(2))+h(t;2A_I(1))$, $h^*(s;2A_I)=h^*(s;2A_I(2))+h^*(s;2A_I(1))$ and $h(t;2A_I)=h(t;2A_C)$.

For Red side, $h(t;2B_I(2))$, $h(t;2B_I(1))$, $h(t;2B_I)$, $h^*(s;2B_I(2))$, $h^*(s;2B_I(1))$ and $h^*(s;2B_I)$ can be similarly defined.

With assumptions (i)-(iv) and (v-I)-(vi-I), we obtain

(32) $P(T;\bar{I},I)=\int_0^T \{p(t;\bar{I},I,0)+p(t;\bar{I},I,1)\}\,dt$

where $p(t;\bar{I},I,0)=h(t;2\bar{A}_I(2),2B_I)+(h(2\bar{A}_I(1),2B_I)*h(2\bar{A}_I,B))(t)$

and $p(t;\bar{I},I,1)=(h(2\bar{A}_I(1),2B_I)*h(2_{AI},\bar{B})*h(\bar{A},B))(t)$
$+(h(2A_I,2\bar{B}_I(1))*h(\bar{A},2B_I)*h(\bar{A},B))(t).$

Hence,

(33) $P^*(s;\bar{I},I)=\dfrac{1}{s}\ \{h^*(s;2\bar{A}_I(2),\ 2B_I)+h^*(s;2\bar{A}_I(1),\ 2B_I)\cdot$
$[h^*(s;2\bar{A}_I,B)+h^*(s;2A_I,\bar{B})\cdot h^*(s;\bar{A},B)]$
$+h^*(s;2A_I,\ 2\bar{B}_I(1))\cdot h^*(s;\bar{A},2B_I)\cdot h^*(s;\bar{A},B)\},$

and

(34) $P(\bar{I},I)=H(2\bar{A}_I(2),2B_I)+H(2\bar{A}_I(1),\ 2B_I)\cdot\{H(2\bar{A}_I,B)$
$+H\ (2A_I,\bar{B})\cdot H(\bar{A},B)\}+H(2A_I,2\bar{B}_I(1))\cdot H(\bar{A},2B_I)\cdot H(\bar{A},B).$

Example-4

As in Example-3, interfiring times are assumed to be negative exponential. Then, by equation (31), we have

(35) $h^*(s;2A_I(2))=\dfrac{r_Ap_A^2}{s+r_A(1-q_A^2)}$, $h^*(s;2A_I(1))=\dfrac{2r_Ap_Aq_A}{s+r_A(1-q_A^2)}$ and

$h^*(s;2A_I)=h^*(s;2A_C)$

where $h^*(s;2A_C)$ is given in equation (20).

With equations (32)-(35) and the procedures leading to equations (14)-(15) and setting $u=r_A(1-q_A^2)$, $v=r_Ap_A$, $x=r_B(1-q_B^2)$, $y=r_Bp_B$ and $z=2r_Bp_Bq_B$, we obtain

$$(36)\quad P(T;\bar{I},I)=C_0+C_1e^{-(u+x)T}+C_2e^{-(u+y)T}+C_3e^{-(v+x)T}+C_4e^{-(v+y)T}$$

where
$$C_0=\frac{v[2u+(y-u)p_A]}{(u+x)(u+y)}+\frac{v^2[2yq_A(v+x)+z(u+y)]}{(u+x)(u+y)(v+y)(v+x)},$$

$$C_1=\frac{v[2u-p_A(2u+x-y)]}{(u+x)(x-y)}+\frac{v^2[2yq_A(u-v)+z(x-y)]}{(u+x)(u-v)(u+x-v-y)(y-x)},$$

$$C_2=\frac{2vq_A[u(u-v)-vy]}{(u+y)(u-v)(y-x)},$$

$$C_3=\frac{v^2z}{(v+x)(u-v)(x-y)},$$

$$C_4=\frac{v^2[2yq_A(x-y)+z(u-v)]}{(v+y)(u-v)(u+x-v-y)(y-x)}.$$

Blue's winning probability $P(\bar{I},I)=C_0$ without time limitation can also be obtained from equations (8), (33) and (35), and rewritten as

$$(37)\quad P(\bar{I},I)=\left(\frac{r_Ap_A^2}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)+\left(\frac{2r_Ap_Aq_A}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)$$
$$\left\{\left(\frac{r_A(1-q_A^2)}{r_A(1-q_A^2)+r_Bp_B}\right)+\left(\frac{r_Bp_B}{r_A(1-q_A^2)+r_Bp_B}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_Bp_B}\right)\right\}$$
$$+\left(\frac{2r_Bp_Bq_B}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_B(1-q_B^2)}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_Bp_B}\right).$$

VII. SQUARE DUEL-5: (I,C)

This duel is similar to the first square duel of Ancker and Williams [2], except that it employs continuous random interfiring times whereas they use fixed intervals between firings.

With arguments similar to sections Ⅳ and Ⅵ and using assumptions in section II, the winning probabilities of both sides are obtained as follows.

For Blue side with strategy (I) against Red with strategy (C), we have

$$(38)\quad P(T;\bar{I},C)=\int_0^T \{p(t;\bar{I},C,0)+p(t;\bar{I},C,1)\}\,dt$$

where

$$p(t;\bar{I},C,0)=h(t;2\bar{A}_I(2),2B_c)+(h(2\bar{A}_I(1),2B_c)*h(2\bar{A}_I,B))(t)$$

and

$$p(t;\bar{I},C,1)=(h(2\bar{A}_I(1),2B_c)*h(2A_I,\bar{B})*h(\bar{A},B))(t)$$
$$+(h(2A_I,2\bar{B}_c)*h(\bar{A},2B_c)*h(\bar{A},B))(t).$$

Hence,

$$(39)\quad P^*(s;\bar{I},C)=\frac{1}{s}\{h^*(s;2\bar{A}_I(2),2B_c)+h^*(s;2\bar{A}_I(1),2B_c)\cdot$$
$$[h^*(s;2\bar{A}_I,B)+h^*(s;2A_I,\bar{B})\cdot h^*(s;\bar{A},B)]$$
$$+h^*(s;2A_I,2\bar{B}_c)\cdot h^*(s;\bar{A},2B_c)\cdot h^*(s;\bar{A},B)\},$$

and

$$(40)\quad P(\bar{I},C)=H(2A_I(2),2B_c)+H(2\bar{A}_I(1),2B_c)\cdot\{H(2\bar{A}_I,B)$$
$$+H(2A_I,\bar{B})\cdot H(\bar{A},B)\}+H(2A_I,2\bar{B}_c)\cdot H(\bar{A},2B_c)\cdot H(\bar{A},B).$$

For Red side,

$$(41)\quad P^*(s;I,\bar{C})=\frac{1}{s}\{h^*(s;2A_I,2\bar{B}_c)\cdot[h^*(s;A,2\bar{B}_c)+h^*(s;\bar{A},2B_c)\cdot$$
$$h^*(s;A,\bar{B})]+h^*(s;2\bar{A}_I(1),2B_c)\cdot h^*(s;2A_I,\bar{B})\cdot h^*(s;A,\bar{B})\},$$

and

$$(42)\quad P(I,\bar{C})=H(2A_I,2\bar{B}_c)\cdot\{H(A,2\bar{B}_c)+H(\bar{A},2B_c)\cdot H(A,\bar{B})\}$$
$$+H(2\bar{A}_I(1),2B_c)\cdot H(2A_I,\bar{B})\cdot H(A,\bar{B}).$$

Example-5

Let interfiring times be negative exponential. Then, we can obtain $P(T;\bar{I},C)$ expressed as equation (36) with u,v,x,y and z replaced by $r_A(1-q_A^2), r_Ap_A, r_B(1-q_B^2), r_Bp_B$ and $r_B(1-q_B^2)$ respectively.

Then. $P(\bar{I},C)$ can be written as

$$(43)\quad P(\bar{I},C)=\left(\frac{r_Ap_A^2}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)+\left(\frac{2r_Ap_Aq_A}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)$$

$$\left[\left(\frac{r_A(1-q_A^2)}{r_A(1-q_A^2)+r_Bp_B}\right)+\left(\frac{r_Bp_B}{r_A(1-q_A^2)+r_Bp_B}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_Bp_B}\right)\right]$$
$$+\left(\frac{r_B(1-q_B^2)}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_B(1-q_B^2)}\right)\left(\frac{r_Ap_A}{r_Ap_A+r_Bp_B}\right).$$

For Red side, we have

$$(44)\quad P(I,\bar{C})=\left(\frac{r_B(1-q_B^2)}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)\left[\left(\frac{r_B(1-q_B^2)}{r_Ap_A+r_B(1-q_B^2)}\right)\right.$$
$$\left.+\left(\frac{r_Ap_A}{r_Ap_A+r_B(1-q_B^2)}\right)\left(\frac{r_Bp_B}{r_Ap_A+r_Bp_B}\right)\right]+\left(\frac{2r_Ap_Aq_A}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)$$
$$\left(\frac{r_Bp_B}{r_A(1-q_A^2)+r_Bp_B}\right)\left(\frac{r_Bp_B}{r_Ap_A+r_Bp_B}\right).$$

To show the superiority of individually seperated strategy (I) against concentrated strategy (C), three measures $D_C(I/C)$, $D_I(I/C)$ and $D(I,C)$ are defined as

$$(45)\quad D_C(I/C)\equiv P(\bar{I},C)-P(\bar{C},C),$$

$$(46)\quad D_I(I/C)\equiv P(\bar{I},I)-P(\bar{C},I),$$

and

$$(47)\quad D(I,C)\equiv P(\bar{I},C)-P(I,\bar{C})$$

respectively. Then, from equations (21), (37) and (43)-(46), we obtain

$$D_C(I/C)=\left(\frac{r_Ap_A^2}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)\left(\frac{r_Bp_B}{r_A(1-q_A^2)+r_Bp_B}\right)\left(\frac{r_Bp_B}{r_Ap_A+r_Bp_B}\right)\geq 0$$

and $D_I(I/C)=D_C(I/C)$ which show that strategy (I) is always better than strategy (C) no matter which of (C) and (I) the opponent side uses.

If $r_A=r_B$, $p_A=p_B=p$, equations (43)-(44) and (47) reduce to

$$P(\bar{I},C)=\frac{2p^2-9p+12}{4(2-p)(3-p)},\quad P(I,\bar{C})=\frac{2p^2-11p+12}{4(2-p)(3-p)}$$

$$\text{and } D(I,C)=\frac{p}{2(2-p)(3-p)}$$

which are plotted in Fig. 3. It shows that $D(I,C)$ increases to $\frac{1}{4}$ as p increases to 1 but decreases to zero as p goes to zero.

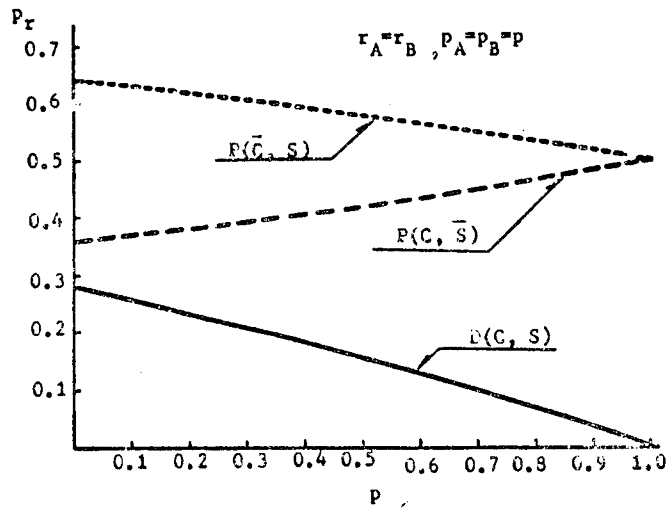


Fig. 3. Seperated Strategy Compared with Concentrated Strategy

VIII. SQUARE DUEL-6: (R,R)

This is the case when both sides are allowed to select initial firing strategy among strategies (I) and (C) at random. Thus, Blue and Red are engaged in one of the four types of square duels, (C,C), (C,I), (I,C) and (I,I) with equal probability.

Hence Blue's winning probability $P(T;\bar{R},R)$ with strategy (R) and limited time T can be expressed as

$$(48)\quad P(T;\bar{R},R)=\frac{1}{4}\{P(T;\bar{C},C)+P(T;\bar{C},I)+P(T;\bar{I},C)+P(T;\bar{I},I)\}$$

where all probabilities inside the bracket may be obtained from equations (17), (32) and (38).

Then, the Laplace transform of equation (48) becomes

$$(49)\quad P^*(s;\bar{R},R)=\left(\frac{1}{2s}\right)\Big\{h^*(s;\ 2\bar{A}_I(2),\ 2B_C)+[h^*(s;2\bar{A}_C,2B_C)$$

$$+h^*(s;\ 2\bar{A}_I(1),\ 2B_C)]\cdot[h^*(s;2\bar{A}_C,B)+h^*(s;2A_C,\bar{B})$$

$$\cdot h^*(s;\ \bar{A},B)]+[h^*(s;2A_C,2\bar{B}_C)+h^*(s;2A_C,2\bar{B}_I(1))]$$

$$\cdot h^*(s;\ \bar{A},2B_C)\cdot h^*(s;\bar{A},B)\Big\}$$

and

(50) $P(\bar{R},R)=\frac{1}{2}\{H(2\bar{A}_I(2),2B_C)+[H(2\bar{A}_C,B_C)+H(2\bar{A}_I(1),2B_C)]$

$\cdot[H(2\bar{A}_C,B)+H(2A_C,\bar{B})\cdot H(\bar{A},B)]+[H(2A_C,2\bar{B}_C)$

$+H(2A_C,2\bar{B}_I(1))]\cdot H(\bar{A},2B_C)\cdot H(\bar{A},B)\}.$

In particular, if p_A is sufficiently small such that p_A^2 is negligible, $h^*(s;A)$, $h^*(s;2A_C)$, $h^*(s;2A_I(2))$ and $h^*(s;2A_I(1))$ in equations (5), (16) and (31) can be written as

$$h^*(s;A)=\frac{p_Af^*(s;A)}{1-q_Af^*(s;A)},\quad h^*(s;2A_C)=h^*(s;2A_I)=\frac{2p_Af^*(s;A)}{1-(1-2p_A)f^*(s;A)},$$

(51) $h^*(s;2A_I(2))=0,\quad h^*(s;2A_I(1))=\frac{2p_Af^*(s;A)}{1-(1-2p_A)f^*(s;A)},$

and thus

(52) $h^*(s;2A_C)=h^*(s;2A_I)=h^*(s;2A_I(1)).$

Then, Blue's winning probabilities in equations (19), (34) and (40) all reduse to

(53) $P(\text{Blue})=H(2\bar{A}_C,2B_C)\cdot[H(2\bar{A}_C,B)+H(2A_C,\bar{B})\cdot H(\bar{A},B)]$

$+H(2A_C,2\bar{B}_C)\cdot H(\bar{A},2B_C)\cdot H(\bar{A},B)$

where $P(\text{Blue})$ is the probability that Blue wins when duel time is unlimited and p_A and p_B have small values. We note that equation(53) implies that strategy (I) has no advantage over strategy (C).

Example-6.

Let interfiring times be negative exponential. Then, by substituting equations (11), (20) and (35) into equations (49)-(50), we obtain

$$P^*(s;\bar{R},R)=\frac{1}{2s}\Big\{\Big(\frac{r_Ap_A^2}{s+r_A(1-q_A^2)+r_B(1-q_B^2)}\Big)+\Big[\Big(\frac{r_A(1-q_A^2)}{s+r_A(1-q_A^2)+r_B(1-q_B^2)}\Big)$$

$$+\Big(\frac{2r_Ap_Aq_A}{s+r_A(1-q_A^2)+r_B(1-q_B^2)}\Big)\Big]\cdot\Big[\Big(\frac{r_A(1-q_A^2)}{s+r_A(1-q_A^2)+r_Bp_B}\Big)$$

$$+\Big(\frac{r_Bp_B}{s+r_A(1-q_A^2)+r_Bp_B}\Big)\Big(\frac{r_Ap_A}{s+r_Ap_A+r_Bp_B}\Big)\Big]$$

$$+\Big[\Big(\frac{r_B(1-q_B^2)}{r_A(1-q_A^2)+r_B(1-q_B^2)}\Big)+\Big(\frac{2r_Bp_Bq_B}{s+r_A(1-q_A^2)+r_B(1-q_B^2)}\Big)\Big]$$

$$\left(\frac{r_A p_A}{s+r_A p_A+r_B(1-q_B^2)}\right)\left(\frac{r_A p_A}{s+r_A p_A+r_B p_B}\right)\Big\},$$

and

$$(54)\quad P(\bar{R},R)=\frac{1}{2}\Big\{\left(\frac{r_A p_A^2}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)+\Big[\left(\frac{r_A(1-q_A^2)}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)$$

$$+\left(\frac{2r_A p_A q_A}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)\Big]\cdot\Big[\left(\frac{r_A(1-q_A^2)}{r_A(1-q_A^2)+r_B p_B}\right)$$

$$+\left(\frac{r_B p_B}{r_A(1-q_A^2)+r_B p_B}\right)\left(\frac{r_A p_A}{r_A p_A+r_B p_B}\right)\Big]+\Big[\left(\frac{r_B(1-q_B^2)}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)$$

$$+\left(\frac{2r_B p_B q_B}{r_A(1-q_A^2)+r_B(1-q_B^2)}\right)\Big]\left(\frac{r_A p_A}{r_A p_A+r_B(1-q_B^2)}\right)\left(\frac{r_A p_A}{r_A p_A+r_B p_B}\right).$$

When p_A and p_B have small values, equations (51)-(52) becomes

$$h^*(s;A)=\frac{r_A p_A}{s+r_A p_A},\quad\text{and}\quad h^*(s;2A_C)=h^*(s;2A_I)=\frac{2r_A p_A}{s+2r_A p_A}.$$

Using these in equation (53), we obtain

$$(55)\quad P(\text{Blue})=\left(\frac{2r_A p_A}{2r_A p_A+2r_B p_B}\right)\Big\{\left(\frac{2r_A p_A}{2r_A p_A+r_B p_B}\right)+\left(\frac{r_B p_B}{2r_A p_A+r_B p_B}\right)$$

$$\left(\frac{r_A p_A}{r_A p_A+r_B p_B}\right)\Big\}+\left(\frac{2r_B p_B}{2r_A p_A+2r_B p_B}\right)\left(\frac{r_A p_A}{r_A p_A+2r_B p_B}\right)\left(\frac{r_A p_A}{r_A p_A+r_B p_B}\right)$$

which coincides with the results of equations(21), (37), (43) and (54) when $1-q_A^2=2p_A$, $1-q_B^2=2p_B$, $p_A^2=0$ and $p_B^2=0$.

IX. CONCLUSIONS

This paper presents general solutions for stochastic square duels with continuous interfiring times and various firing strategies such as standby (S), concentrated (C), seperated (I) and random (R) firings.

Analyses of these square duels with negative exponential interfiring times and equivalent values of rates of fire and single shot kill probabilities reveal three important facts:

i) Strategy (C) is advantageous against the opponent's strategy (S) and the advantage becomes more pronounced for lower values of single shot kill probabilities.

ii) Strategy (I) is always better than strategy (C) no matter which of (C) and (I) the opponent uses and its relative advantage increases to a quarter as single shot kill probabilities increase to one but decreases to zero as they go to zero.

iii) However, strategy (I) has no advantage over strategy (C) for small values of single shot kill probabilities.

In this paper, square duels with strategies (C) and (I) are based on the assumptions that duelists are homogeneous and both duelists of one side fire simultaneously. The problem of relaxing these assumptions and extension of square (2×2) duels to more general $(m\times n)$ duels are now being investigated.

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