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Optimal Control of Delay-Differential System under Function Target Condition

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Abstract

The problem of optimally controlling a time-delay control system to a function as the final target is inverstigated. Necessary conditions are presented in the form of Pontryagin's maximum principle, and it is further shown that they are also sufficient for linear systems with a convex cost functional. Several examples are given to illustrate the results.

1. INTRODUCTION

In many control systems, the controller action is derived from the decision based on the delayed state information. In these systems, controlled quantities are often transmitted over a long distance and hence the control action takes its effect after some time. In a typical cold rolling mill where steel plate is rolled down through several rollers, for example, the main press roller and the thickness-sensing gage are physically seperated by a considerable distance. Since the roller speed is rather slow, a time-dalay (spacing/speed) is introduced to the controller-actuator regarding the information on the steel plate thickness.

Time-delay in the state-information often causes an undesirable system performance such as oscillation, lengthy settling time or even breakdown. The time-delay effect can be found not only in the industrial processes but also in other engineering systems such as rocket control systems, nuclear reactor control processes, or in physiological and socioeconomic systems. These time-delay systems are mathematically modeled by delay differential equa tions [15].

Control problems involving the time-delay systems have been investigated by many researchers (see the references of [1] and [5]). Before 1970, the problem was usually formulated in a fashion similar to that of ordinary differential system. Thus the usual optimal control problem investigated was that of finding a controller which steers the response of a time-delay system from a given initial function to a final point (in a finite dimensional space) while minimizing a given cost functional. Note that, in systems without time delays, once the target point is reached, it is usually possible to stay at that state thereafter. For example, once the origin of the state space is reached in a linear control system, the system can be kept at the origin forever by applying a zero control function. However, when there is a time-delay in the system, reaching a final point at a given time does not guarantes that the system can be kept at that state thereafter. It is in fact not very difficult to construct an example which reveals the contrary result, *i.e.*, the system state reaches the origin but its speed is nonzere due to the delay effect so that the system can not stay at the origin for some time. In most of the practical control systems, however, the objective is to change the present system state to a new state and to keep it in the new state. In rendezvous of two or more controlled systems, the task is carried out in practice only when the rendezvous state is maintained for somenonzero duration. Therefore the classical formulation in which the control sche-

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me is to steer the time delay system to a point in R^n has been found to be practically inappropriate.

A more realistic approach for the control of time-delay systems is to steer the response of the system to a final function as the target. In order to regulate, for example, a time-delay system $\dot{x}(t)=f(x(t), x(t-h), u(t))$ to an equilibrium state in a finite time T, it is sufficient and necessary to drive the response to a final function $x(t)=\Psi(t)$ such that $f(\Psi(t), \Psi(t-h), u(t))=0$ on the non-zero interval $[T-h, T]$. From a system-theoretic point of view. the state should be given as an n-vector function for the class of time-delay systems. Various examples and justification can be found in Reference [5].

In this paper, problems of optimally controlling a system with delays both in state and control variables are investigated. Necessary and sufficient conditions are derived and several examples are solved to illustrate the effectiveness of the result.

Throughout the paper, the norm of a vector b in R^j (the j-dimensional Euclidean space) is denoted by $|b|$. Column and row vectors are not distinguished unless there is a possibility of confusion. For a given matrix, B, the B^T and B^{-1} denote the transpose and the inverse of B, respectively. Given a compact interval I and a positive integer j, $C^j(I)$ denotes the linear space of j-vector continuous functions on I with sup norm topology, and $L^j_\infty(I)$ denotes the linear space of essentially bounded j-vector functions on I with essential sup norm topology. $z(\cdot)$, or simply z in some cases, denotes the function $z(t)$ on an interval when it is regarded as an element of a function space. I/I_1 denotes the set of points in I but not in I_1. The partial derivatives are denoted by subscripts, *e.g.*, $f_x(x_0, y, u, v, t) = \frac{\partial}{\partial x} f(x, y, u, v, t)|_{x=x_0}$

2. OPTIMAL CONTROL OF DELAY SYSTEM TO FUNCTION-TARGET

When a time delay is involved in a control system, it is more realistic to steer the system response to a final function as a target rather than to final point in R^n-space. The problem of optimally controlling a time-delay system to a final function has been studied by several investigators in references [3, 4, 10]. In [3] and [10] no magnitude constraints were imposed on the control variable. In [4] the necessary condition obtained was in an integral form of maximum principle and the nontriviality of the adjoint solution was not guaranteed.

In this section, the control systems with delays both in the state variable and in the control variable are investigated. A necessary condition for an optimal control is provided in the form of a pointwise maximum principle with a nontrivial adjoint solution. This result is applicable for the cases where there are magnitude constraints on the control variables. It is further shown that this necessary condition is also sufficient for certain linear systems with convex cost functionals. These results are then extended to a more general target set in function space.

2.1. Function-Target Problem

Consider a control system with a time delay

$\dot{x}(t)=f(x(t), x(t-h), u(t), u(t-h), t)$,

where x is an n-vector state variable, u is an m-vector control variable, and $0< h < \infty$ is the time delay. The vector function $f(x,y,u,v,t)=(f^1(x,y,u,v,t), f^2(x,y,u,v,t),\cdots,f^n(x,y,u,v,t))$, and its derivatives $f_x(x,y,u,v,t)$, $f_y(x,y,u,v,t)$, $f_u(x,y,u,v,t)$ and $f_v(x,y,u,v,t)$ are continuous in x,y,u,v and t. The initial condition at $t=t_0$ is given by $x(t)=\phi(t)$ on $[t_0-h, t_0]$ and $u(t)=u_0(t)$ on $[t_0-h, t_0)$ where $\phi(t)$ is a given continuous initial function for the state and $u_0(t)$ is a given measurable initial function for the control variable. Let the control restraint set Ω be given by $\Omega=\{u | u\in R^m, q^i(u)\leq 0, i=1,2,\cdots,r\}$, where $q^i(u)$ is a smooth scalar function for each i. A control function $u(t)$ on $[t_0, t_1]$ is called admissible if it is measurable, essentially bounded, and $u(t)\in\Omega$ *a.e.* on $[t_0, t_1]$.

The problem is to find an admissible control $u(t)$ on a given interval $[t_0, t_1]$ with $t_1>t_0+2h$ which steers the corresponding system response from a given initial function $x(t)=\phi(t)$ on $[t_0-h, t_0]$ to a given smooth final function $x(t)=\psi(t)$ on $[t_1-h, t_1]$ while minimizing the cost functional

$$C(u(\cdot))=\int_{t0}^{t1} f^0(x(t),\ x(t-h),\ u(t),\ u(t-h),\ t)dt$$

Here, the scalar valued function $f^0(x,y,u,v,t)$, and the derivatives $f_x^0(x,y,u,v,t)$, $f_y^0(x,y,u,v,t)$, $f_u^0(x,y,u,v,t)$ and $f_v^0(x,y,u,v,t)$ are continuous in x,y,u,v and t.

2.2 Necessary Condition

For an admissible control $u(t)$ on $[t_0,\ t_1]$, let $A_0(u(\cdot))$ denote the cone in $L_\infty^m[t_0,\ t_1]$ defined by $A_0(u(\cdot))=\{\alpha\delta u|\alpha>0,\ \delta u\in L_\infty^m(I),\ \delta u(t)=0$ a.e. on $[t_0,\ t_1-2h]$, $q^i(u(t))+q_u^i(u(t))\ \delta u(t)<0$ a.e. on $[t_1-2h,\ t_1]$, $i=1,2,\cdots,r\}$. Let $\bar{x}(t)$ be the response of the system for an admissible control $\bar{u}(t)$. For each $t\in[t_1-h,\ t_1]$, let $N(t,\ u(t))$ denote the $n\times n$ matrix

$$N(t,\ u(t))=f_u(\bar{x}(t),\ \bar{x}(t-h),\ u(t),\ u(t-h),\ t)\ [f_u(\bar{x}(t),\ \bar{x}(t-h),\ u(t),\ u(t-h),\ t]^T +f_v(\bar{x}(t),\ \bar{x}(t-h),\ u(t),\ u(t-h),\ t)\ [f_v(\bar{x}(t),\ \bar{x}(t-h),\ u(t),\ u(t-h),\ t]^T.$$

Define the $n\times n$ matrix function $N(t,\ u(t))$ on $[t_1-h,\ t_1]$ by $\tilde{N}(t,\ u(t))=N(t,\ u(t))^{-1}$ if $\tilde{N}(t,\ u(t))$ is nonsingular at t and $\tilde{N}(t,\ u(t))=0$ otherwise. An admissible control $u(t)$ is called regular with respect to the response $\bar{x}(t)$ if

(i) $f(\bar{x}(t),\ \bar{x}(t-h),\ u(t),\ u(t-h),\ t)=\dot{\phi}(t)$ a.e. on $[t_1-h,\ t_1]$,

(ii) $N(t,u(t))$ is nonsingular a.e. on $[t_1-h,\ t_1]$ and each component of the matrix function $\tilde{N}(t,\ u(t))$ on $[t_1-h,\ t_1]$ is in $L_\infty[t_1-h,\ t_1]$.

(iii) For any $z\in L_\infty^n[t_1-h,\ t_1]$, there exists $\delta u_z\in A_0(u(\cdot))$ such that

$$z(t)=f_u(\bar{x}(t),\ \bar{x}(t-h),\ u(t),\ u(t-h),\ t)\delta u_z(t) +f_v(\bar{x}(t)),\ \bar{x}(t-h),\ u(t),\ u(t-h),\ t)\delta u_z(t-h)$$
a.e. on $[t_1-h,\ t_1]$.

For each $(x,y)\in R^n\times R^n$, $v\in\Omega$ and $t\in[t_1-h,\ t_1]$, let $W_1(x,y,v,t)$ be the set of all $u\in\Omega$ such that

(i) $f(x,y,u,v,t)=\dot{\phi}(t)$,

(ii) $f_u(x,y,u,v,t)\ f_u(x,y,u,v,t)^T+f_v(x,y,u,v,t)\ f_v(x,y,u,v,t)$ is nonsingular,

(iii) $\{f_u(x,y,u,v,t)\Delta u+f_v(x,y,u,v,t)\Delta v|\Delta v\in R^m$, $q_u^j(u)\Delta u<0$ for each $j=1,\cdots,r$ such that $q^j(u)=0$, $\Delta v\in R^m$, $q_u^j(v)\Delta v<0$ for each j such that $q^j(v)=0\}=R^n$.

For each $(x,y)\in R^n\times R^n$, $u\in\Omega$, and $t\in[t_1-h,\ t_1]$, let $W_2(x,y,u,t)$ be the set of all $v\in\Omega$ such that

(i) $f(x,y,u,v,t)=\dot{\phi}(t)$

(ii) $f_u(x,y,u,v,t)f_u(x,y,u,v,t)^T+f_v(x,y,u,v,t)\ f_v(x,y,u,v,t)^T$ is nonsingular,

(iii) $\{f_u(x,y,u,v,t)\Delta u+f_v(x,y,u,v,t)\Delta v|\Delta u\in R^m$, $q_u^j(u)\Delta u<0$ for each $j=1,\cdots,r$ such that $q^j(u)=0$, $\Delta v\in R^m$, $q_u^j(v)\Delta v<0$ for each j such that $q^j(v)=0\}=R^n$.

Let $f=(f^0,\ f)$ and $q=(q^1,\cdots,q^r)$. Also, let $V_1^*(t)=W_1(x^*(t),\ x^*(t-h),\ u^*(t-h),\ t)$, and $V_2^*(t)=W_2^*(x^*(t+h),\ x^*(t),\ u^*(t+h),\ t+h)$.

Theorem 2.1. NECESSARY CONDITION (Maximum Principle). Let $u^*(t)$ with the response $x^*(t)$ be an optimal control. Let $u^*(t)$ be regular with respect to $x^*(t)$. Then there exist functions $\nu=(\nu^1,\nu^2,\cdots,\nu^n)\in L_\infty^n[t_1-h,\ t_1]$, $\mu=(\mu^1,\mu^2,\cdots,\mu^r)\in L_\infty^r[t_1-2h,\ t_1]$, and an $(n+1)$-vector absolutely continuous function $n(t)=(n^0(t),\ n(t))=(n^0(t),\ n^1(t),\cdots,n^n(t))$ on $[t_0,\ t_1]$ such that

(i) $\hat{n}(t_1)\neq 0$

(ii) $\dot{n}^0(t)=0,\ t\in[t_0,\ t_1],\ n^0(t_1)=n^0\leq 0$

(iii)
$$-\dot{n}(t)=n^0[f_x^0(x^*(t),\ x^*(t-h),\ u^*(t),\ u^*(t-h),\ t)+f_y^0(x^*(t+h),\ x^*(t),\ u^*(t+h),\ u^*(t),\ t+h)]+n(t)f_x(x^*(t),\ x^*(t-h),\ u^*(t),\ u^*(t-h),\ t)+n(t+h)f_y(x^*(t+h),\ x^*(t),\ u^*(t+h),\ u^*(t),\ t+h)$$
a.e. on $[t_0,\ t_1-2h]$,

$$-\dot{n}(t)=n^0[f_x^0(x^*(t),x^*(t-h),u^*(t),u^*(t-h),t)+f_y^0(x^*(t+h),\ x^*(t),u^*(t+h),u^*(t),t+h)]+n(t)f_x(x^*(t),x^*(t-h),u^*(t),u^*(t-h),t)+[n(t+h)+\nu(t+h)]f_y(x^*(t+h),\ x^*(t),\ u^*(t+h),\ u^*(t),\ t+h)$$
a.e. on $[t_1-2h,\ t_1-h]$,

$$-\dot{n}(t)=n^0f_x^0(x^*(t),\ x^*(t-h),\ u^*(t),\ u^*(t-h),\ t)+[n(t)+\nu(t)]f_x(x^*(t),\ x^*(t-h),\ u^*(t),\ u^*(t-h),\ t)$$
a.e. on $[t_1-h,\ t_1]$.

(iva)
$$\hat{n}(t)\hat{f}(x^*(t),\ x^*(t-h),\ u^*(t),\ u^*(t-h),\ t)+\hat{n}(t+h)\hat{f}(x^*(t+h),\ x^*(t),\ u^*(t+h),\ u^*(t),\ t+h)$$
$$=\max_{u\in\Omega}[\hat{n}(t)\ \hat{f}(x^*(t),\ x^*(t-h),u,u^*(t-h),t)+\hat{n}(t+h)\hat{f}(x^*(t+h),\ x^*(t),\ u^*(t+h),\ u,$$

$t+h)]$

a.e. on $[t_0, t_1-2h]$,

(ivb) $\hat{n}(t)\hat{f}(x^*(t), x^*(t-h), u^*(t), u^*(t-h), t)$
$+\hat{n}(t+h)\hat{f}(x^*(t+h), x^*(t), u^*(t+h), u^*(t), t+h)$
$=\max_{v\in V_2^*(t)} [\hat{n}(t)\hat{f}(x^*(t), x^*(t-h), v, u^*(t-h), t)$
$+\hat{n}(t+h)\hat{f}(x^*(t+h), x^*(t), u^*(t+h), v, t+h)]$

a.e. on $[t_1-2h, t_1-h]$,

(ivc) $\hat{n}(t)\hat{f}(x^*(t), x^*(t-h), u^*(t), u^*(t-h), t)$
$=\max_{u\in V_1^*(t)} [\hat{n}(t)\hat{f}(x^*(t), x^*(t-h), u, u^*(t-h), t)$

a.e. on $[t_1-h, t_1]$.

(v) $\mu(t)q_u(u^*(t))+\nu(t+h)f_v(x^*(t+h), x^*(t), u^*(t+h), u^*(t), t+h)$
$+\hat{n}(t)\hat{f}_u(x^*(t), x^*(t-h), u^*(t), u^*(t-h), t)$
$+n(t+h)\hat{f}_v(x^*(t+h), x^*(t), u^*(t+h), u^*(t), t+h)=0$,

a.e. on $[t_1-2h, t_1-h]$,

$\mu(t)q_u(u^*(t))+\nu(t)f_u(x^*(t), x^*(t-h), u^*(t), u^*(t-h), t)$
$+\hat{n}(t)\hat{f}_u(x^*(t), x^*(t-h), u^*(t), u^*(t-h), t)=0$,

a.e. on $[t_1-h, t_1]$;

(vi) $\mu^i(t)\leq 0$ a.e. on $[t_1-2h, t_1]$, $i=1,2,\cdots,r$,

(vii) $\mu^i(t)q^i(u^*(t))=0$ a.e. on $[t_1-2h, t_1]$, $i=1,2,\cdots,r$.

To prove the above necessary condition, let $\hat{x}=(x^0, x)\in R^{n+1}$, $\hat{y}=(y^0, y)\in R^{n+1}$, and $\hat{f}(\hat{x}, \hat{y}, u, v, t)=(f^0(x,y,u,v,t), f(x,y,u,v,t))$. Define scalar functions $p^i(\hat{x},\hat{y},u,v,t)$, $i=1,\cdots,n$, and $\chi^i(x)$, $i=0,1,\cdots,n$, by

$$p^i(\hat{x},\hat{y},u,v,t)=f^i(x,y,u,v,t)-\dot{\psi}^i(t), i=1,2,\cdots,n$$
$$\chi^i(\hat{x})=x^i-\psi^i(t_1), \ i=1,2,\cdots,n$$
$$\chi^0(\hat{x})=x^0.$$

Given an admissible control $u(t)$ with respense $x(t)$, let $x^0(t)$ be an absolutely continuous function on I such that $x^0(t_0)=0$ and

$$\dot{x}^0(t)=f^0(x(t), x(t-h), u(t), u(t-h), t)$$
a.e. on I.

Then the optimal control problem is equivalent to the following problem: Find a bounded measurable control $u(t)$ on I with response $\hat{x}(t)=(x^0(t), x(t))$ such that

$\dot{\hat{x}}(t)=\hat{f}(\hat{x}(t), \hat{x}(t-h), u(t), u(t-h), t)$ a.e. on I
$\hat{x}(t)=(0, \phi(t))$ and $u(t)=u_0(t)$, on $[t_0-h, t_0]$
$\chi^i(\hat{x}(t_1))=0$, $i=1,2,\cdots,n$
$p^i(\hat{x}(t), \hat{x}(t-h), u(t), u(t-h), t)=0$ a.e. on I_1, $i=1,\cdots,n$
$q^i(u(t))\leq 0$ a.e. on I, $i=1,2,\cdots,r$

and $\chi^0(\hat{x}(t_1))$ is minimun. The theorem then can be proved by using an argument similar to that in ref. [11].

2.3. Sufficient Condition

In this section, the system will be restricted to a linear control system with a convex control restraint set and a convex cost functional.

Consider the linear system

$$\dot{x}(t)=A_0(t)x(t)+A_1(t)x(t-h)+B_0(t)u(t)+B_1(t)u(t-h)$$

where the coefficient matrices are continuous on $[t_0, t_1]$. The convex control restraint set Ω is given by

$$\Omega=\{u|u\in R^m, q^i(u)\leq 0, i=1,2,\cdots,r\}$$

where $q^i(u)$ is convex and continuously differentiable in u for each i. The cost functional $C(u(\cdot))$ is given by

$$C(u(\cdot))=\int_{t_0}^{t_1}[s^0(x(t), t)+c^0(u(t), t)]dt$$

where the scalar functions $s^0(x,t)$ and $c^0(u,t)$ are continuously differentiable and convex in x and u, respectively, for each t, and are continuous in t for each x and u.

Let

$$f(x,y,u,v,t)=A_0(t)x+A_1(t)y+B_0(t)u+B_1(t)v$$
$$f^0(x,y,u,v,t)=s^0(x,t)+c^0(u,t)$$

and let $x^*(t)$ be the response of the system corresponding to an admissible control $u^*(t)$ with $u^*(t)=u_0(t)$, $t_0-h\leq t\leq t_0$.

Theorem 2.2. SUFFICIENT CONDITION. Suppose $x^*(t)=\phi(t)$ $t-h\leq t\leq t_0$, and $x^*(t)=\psi(t)$, $t_1-h\leq t_1\leq t_1$. Also, suppose that there exist functions $\nu=(\nu^1,\nu^2,\cdots,\nu^n)\in L_\infty^n[t_1-h, t_1]$, $\mu=(\mu^1,\mu^2,\cdots\mu^r)\in L_\infty^r]t-2h, t_1]$, and an absolutely continuous function $\hat{n}(t)=(n^0(t), n(t))$ on $[t_0, t_1]$ with $n^0(t_1)=n^0<0$ such that the conditions (ii)—(viia) in Theorem 2.1 are satisfied. The $u^*(t)$ is an optimal control.

To prove the above sufficient condition, let $u(t)$ be an admiesible control which steers the system response $x(t)$ from the initial function $x(t)=\phi(t)$

on $[t_0-h, t_0]$ to the final function $x(t)=\psi(t)$ on $[t_1-h, t_1]$. Let

$$\gamma=n^0\int_{t0}^{t1}[f^0(x(t), x(t-h), u(t), u(t-h), t) -f^0(x^*(t), x^*(t-h), u^*(t), u^*(t-h), t)]dt$$

Then, it suffices to show that $\gamma\leq 0$ for any admissible control $u(t)$.

$$x(t)-x^*(t) =\begin{cases} 0, & t_0-h\leq t\leq t_0 \\ \int_{t0}^{t}[A_0(s)(x(s)-x^*(s))+A_1(s)(x(s-h)-x^*(s-h))+B_0(s)(u(s)-u^*(s))+B_1(s)(u(s-h)-u^*(s-h)]ds. \end{cases}$$

Since $n(t)$ is absolutely continuous on $[t_0, t_1]$,

$$\gamma=n^0\int_{t0}^{t1}[f^0(x(t), x(t-h), u(t), u(t-h), t)-f^0(x^*(t), x^*(t-h), u^*(t), u^*(t-h), t)]dt$$
$$+\int_{t0}^{t1}dn(t)\cdot[x(t)-x^*(t)]$$
$$-\int_{t0}^{t1}dn(t)\int_{t0}^{t}[A_0(s)(x(s)-x^*(s))+A_1(s)(x(s-h)-x^*(s-h))+B_0(s)(u(s)-u^*(s))+B_1(s)(u(s-h)-u^*(s-h))]ds$$
$$=n^0\int_{t0}^{t1}[f^0(x(t), x(t-h), u(t), u(t-h), t)-f^0(x^*(t), x^*(t-h), u^*(t), u^*(t-h)]dt$$
$$+\int_{t0}^{t1}\dot n(t)[x(t)-x^*(t)]dt$$
$$+\int_{t0}^{t1}n(t)[A_0(t)(x(t)-x^*(t))+A_1(t)(x(t-h)-x^*(t-h))+B_0(t)(u(t)-u^*(t))+B_1(t)(u(t-h)-u^*(t-h))]dt$$
$$-n(t_1)(x(t_1)-x^*(t_1))$$

Note that $x(t_1)=x^*(t_1)=\psi(t_1)$. Therefore, by the condition (iii),

$$\gamma=n^0\int_{t0}^{t1}[f^0(x(t), x(t-h), u(t), u(t-h), t)-f^0(x^*(t), x^*(t-h), u^*(t), u^*(t-h), t)]dt$$
$$+\int_{t0}^{t1-h}[\dot n(t)+n(t)A_0(t)+n(t+h)A_1(t+h)](x(t)-x^*(t))dt$$
$$+\int_{t1-h}^{t1}[\dot n(t)+n(t)A_0(t)](x^*(t))dt$$
$$+\int_{t0}^{t1-h}[n(t)B_0(t)+n(t+h)B_1(t+h)](u(t)-u^*(t))dt$$
$$+\int_{t1-h}^{t1}n(t)B_0(t)[u(t)-u^*(t)]dt$$
$$=n^0\int_{t0}^{t1}[s^0(x(t), t)-s^0(x^*(t), t)]dt$$
$$-n^0\int_{t0}^{t1}s_x^0(x^*(t), t)(x(t)-x^*(t))dt$$
$$-\int_{t1-2h}^{t1-h}\nu(t+h)A_1(t+h)(x(t)-x^*(t))dt$$
$$-\int_{t1-h}^{t1}\nu(t)A_0(t)(x(t)-x^*(t))dt$$
$$+n^0\int_{t0}^{t1}[C^0(u(t), t)-C^0(u^*(t), t)]dt$$
$$+\int_{t0}^{t1-h}[n(t)B_0(t)+n(t+h)B_1(t+h)](u(t)-u^*(t))dt$$
$$+\int_{t1-h}^{t1}n(t)B_0(t)[u(t)-u^*(t)]dt$$

Now, $x(t)=\psi(t)=x^*(t)$ on $[t_1-h, t_1]$ implies $\dot x(t)-\dot x^*(t)=0$ a.e. on $[t_1-h, t_1]$, that is,

$$A_0(t)(x(t)-x^*(t))+A_1(t)(x(t-h)-x^*(t-h))+B_0(t)(u(t)-u^*(t))+B_1(t)(u(t-h)-u^*(t-h))=0 \quad \text{a.e. on } [t_1-h, t_1].$$

Also, since $s^0(x, t)$ is convex in x,

$$n^0[s^0(x(t), t)-s^0(x^*(t), t)-s_x^0(x^*(t), t)(x(t)-x^*(t))]\leq 0 \quad \text{on } [t_0, t_1].$$

Therefore,

$$\gamma\leq n^0\int_{t0}^{t1}[C^0(u(t), t)-C^0(u^*(t), t)]dt$$
$$+\int_{t1-h}^{t1}\nu(t)\{B_0(t)[u(t)-u^*(t)]+B_1(t)[u(t-h)-u^*(t-h)]\}dt$$
$$+\int_{t0}^{t1-h}[n(t)B_0(t)+n(t+h)B_1(t+h)(u(t)-u^*(t))]dt$$
$$+\int_{t1-h}^{t1}n(t)B_0(t)[u(t)-u^*(t)]dt$$

From the condition (iv),

$$\int_{t0}^{t1-2h}\{n^0C(u(t), t)+[n(t)B_0(t)+n(t+h)B_1(t+h)]u(t)\}dt$$
$$\leq\int_{t0}^{t1-2h}\{n^0C(u^*(t), t)+[n(t)B_0(t)+n(t+h)B_1(t+h)]u^*(t)\}dt$$

Hence,

$$\gamma\leq n^0\int_{t1-2h}^{t1}[C^0(u(t), t)-C^0(u^*(t), t)]dt$$
$$+\int_{t1-2h}^{t1-h}[n(t)B_0(t)+n(t+h)B_1(t+h)](u(t)-u^*(t))dt$$
$$+\int_{t1-h}^{t1}n(t)B_0(t)[u(t)-u^*(t)]dt$$
$$+\int_{t1-2h}^{t1-h}\nu(t+h)B_1(t+h)[u(t)-u^*(t)]dt$$

$$+\int_{t1-h}^{t1}\nu(t)B_0(t)[u(t)-u^*(t)]dt$$

By using condition (v),

$$\gamma\leq n^0\int_{t1-2h}^{t1}[C^0(u(t),\ t)-C^0(u^*(t),\ t)]dt$$

$$-n^0\int_{t1-2h}^{t1}C_u^0(u^*(t),\ t)(u(t)-u^*(t))dt$$

$$-\int_{t1-2h}^{t1}\mu(t)q_u(u^*(t))(u(t)-u^*(t))dt$$

Therefore, from the convexity of C^0 and q, conditions (vi) and (vii), $\gamma\leq 0$. This completes the proof.

2.4 Examples

Example 1. Consider a second order system

$$\dot{x}^1(t)=-x^1(t-1)+2x^2(t)+u(t),\quad 0\leq t\leq 3$$

$$\dot{x}^2(t)=-2x^2(t)+u(t-1),\qquad 0\leq t\leq 3$$

The initial functions are

$$\left.\begin{aligned}\phi^1(t)&=1\\ \phi^2(t)&=0\\ u_0(t)&=0\end{aligned}\right\},\quad -1\leq t\leq 0.$$

The problem is to find a measurable controller $u(t)$ on [0, 3] such that

$$|u(t)|\leq 1,\quad 0\leq t\leq 3$$

and the system response is steered to the zero function, *i.e.*,

$$\left.\begin{aligned}\psi^1(t)&=0\\ \psi^2(t)&=0\end{aligned}\right\}\quad 2\leq t\leq 3,$$

while minimizing the cost-functional

$$C(u(\cdot))=\int_0^3 x^2(t)dt$$

Let $q(u)=(u)^2-1$, and let $u^*(t)$ with the response $x^*(t)$ be an optimal control. Suppose $u^*(t)$ is regular with respect to $x^*(t)$. Then there exist a constant n^0, functions $\nu\in L_\infty^2[2,\ 3]$, $\mu\in L_\infty[1,\ 3]$ and absolutely continuous function $n(t)=(n^1(t),\ n^2(t))$ on [0, 3] such that the conditions (i) through(vii) of Theorem 2.1 are satisfied. Using the conditions (v) through (vii) and the regularity assumption one can easily deduce that $\mu(t)=0$ a.e. on [1, 3] and

$$n^1(t)+[n^2(t+h)+\nu^2(t+h)]=0,\quad \text{a.e. on } [1,\ 2]$$

$$n^2(t)+\nu^1(t)=0,\quad \text{a.e. on } [2,\ 3].$$

It then follows from the solution of the adjoint equation and the maximum condition (iv) that

$$u^*(t)=\begin{cases}\operatorname{sgn}\{(b^1-1/2)+b^1t+(b^2-5b^1+5/2)\\ \quad e^{2(t-1)}\}, & \text{a.e. on } [0,\ 1]\\ 2x^{*2}(t+1), & \text{a.e. on } [1,\ 2]\\ x^{*1}(t-1)-2x^{*2}(t) & \text{a.e. on } [2,\ 3]\end{cases}$$

where b^1 and b^2 are constants.

After a lengthy but straight forward computation, one finds that, among the control functions of the form obtained above, the following is the only control function which steers the response to the given final function.

$$u^*(t)=\begin{cases}-1, & 0\leq t\leq w_1;\\ +1, & w_1<t<w_2;\\ -1, & w_2<t\leq 1;\\ 0, & 1<t\leq 2;\\ x^{*1}(t-1), & 2<t\leq 3;\end{cases}\qquad \text{(see below).}$$

The corresponding response is given by

$$x^{*1}(t)=\begin{cases}-2t+1, & 0\leq t\leq w_1\\ -2w_1+1, & w_1\leq t\leq w_2\\ -2(t-w_2)-2w_1+1, & w_2\leq t\leq 1\\ 2+2(w_2-w_1)+t^2-4t+\frac{1}{2}(1-e^{-2(t-1)}), & 1\leq t\leq 1+w_1\\ 2w_1[t-(1+w_1)]+\frac{1}{2}(e^{-2w_1}-2)\\ \quad [1-e^{-2(t-(1+w_1))}]+x^{*1}(1+w_1), & 1+w_1\leq t\leq 1+w_2\\ 2[t^2-(1+w_2)^2]-[2(w_2-w_1)+4](t-(1+w_2)]\\ \quad +[1+\frac{1}{2}(e^{-2w_1}-2)e^{-2(w_2-w_1)}](1-e^{-2(t-(1+w_2))})\\ \quad +x^{*1}(1+w_2), & 1+w_2\leq t\leq 2\\ 0, & 2\leq t\leq 3.\end{cases}$$

$$x^{*2}(t)=\begin{cases}0, & 0\leq t\leq 1;\\ \frac{1}{2}[e^{-2(t-1)}-1], & 1\leq t\leq 1+w_1;\\ \frac{1}{2}+\frac{1}{2}(e^{-2w_1}-2)e^{-2(t-(1+w_1))}, & 1+w_1\leq t\leq 1+w_2;\\ -\frac{1}{2}+[1+\frac{1}{2}(e^{-2w_1}-2)e^{-2(w_2-w_1)}]\\ \quad e^{-2(t-(1+w_2))}, & 1+w_2\leq t\leq 2;\\ 0, & 2\leq t\leq 3,\end{cases}$$

where

$$w_1=\sqrt{2}-1,$$

and

$$w_2=\frac{1}{2}\log\left(\frac{e^2+2e^{2w_1}-1}{2}\right)\cong 0.85.$$

Note that $|x^{*1}(t)|<1$ on [1, 2]. Therefore, $u^*(t)$ satisfies the regularity condition with respect to $x^*(t)$.

Now, by applying the sufficiency result, it is concluded that $u^*(t)$ on [0, 3] obtained above is an optimal controller.

Example 2. In this example, a time-optimal control problem is solved for a single input-single output delay system

$$\dot{x}(t)=-x(t-1)+u(t),\quad t\geq 0.$$

Given the constraint on the control variable as

$|u(t)|\leq 1, \quad t\geq 0,$

it is required to find a measurable control $u^*(t)$ which steers the system in the minimum time t^* from the initial function $\phi(t)=2, \quad t\in[-1, 0]$, to the zero final function $\phi(t)=0, \quad t\in]t^*-1, t^*]$.

To solve the problem, consider a family of fixed terminal-time problems, each of which, denotod as $(P-w)$, reads as follows:

$(P-w)$ Find a pair $(x(\cdot), u(\cdot))\in C^1[-1, w] \times L_\infty^1[0, w]$

such that

(i) $x(\cdot)$ is absolutely continuous on $[0, w]$

(ii) $\dot{x}(t)=-x(t-1)+u(t)$, a.e. on $[0, w]$

(iii) $x(t)=2, \quad t\in[-1, 0]$

(iv) $x(w)=0$

(v) $-x(t-1)+u(t)=0$, a.e. on $[w-1, w]$

(vi) $q(u(t))=u(t))^2-1\leq 0$, a.e. on $[0, w]$

The time-optimal control problem is then equivalent to finding the least number $w=t^*$ such that the problem $(P-t^*)$ has a solution. Note that for each $w>0$, the problem $(P-w)$ is not an optimal control problem.

First observe that, for any $w>0$, the constraints (ii), (iii) and (vi) in $(P-w)$ imply that for each admissible control $u(\cdot)$, the corresponding trajectory $x(\cdot)$ satisfies $x(0)=2$ and $-3\leq\dot{x}(t)\leq -1$ for almost all $t\in[0, 1]$. Further, on $[0, 1]$, the set of attainability, denoted as $A(t)$, is given by

$A(t)=\{x\in R^1: \ -3t+2\leq x\leq -t+2\}, \quad t\in[0, 1].$

Secondly, let t^* denote the minimum time for which $(P-t^*)$ has a solution, $(x^*(\cdot), u^*(\cdot))$. Then the condition (v) & (vi) in $(P-w)$ imply that

$|x^*(t)|\leq 1, \quad t^*-2\leq t\leq t^*-1$

From these two observations, it can be readily concluded that $t^*\geq\frac{7}{3}$.

It turns outthat $t^*=\frac{7}{3}$ is the minimum time for which $(P-t^*)$ has a solution. To show this, consider the following optimal control problems $(\widetilde{P-w})$:

$(\widetilde{P-w})$ Find a pair $(x(\cdot), u(\cdot))\in C^1[-1, w]\times L_\infty^1[0, w]$

such that the conditions (i) through (vi) of $(P-w)$ are satisfied and that (vii)

$$\int_0^w u(t)dt\leq\int_0^w u(t)dt$$

for all $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ satisfying the previous conditions (i)~(vi).

Note that if $(x(\cdot), u(\cdot))$ is a solution of the problem $(\widetilde{P-w})$, then it is also a solution of $(P-w)$.

Applying Theorem 2.1 to the problem $(\widetilde{P-w})$ with $w=\frac{7}{3}$, one finds that, if $(x^*(\cdot), u^*(\cdot))\in C^1[-1, \frac{7}{3}]\times L_\infty^1[0, \frac{7}{3}]$ is a solution, then there exists a number b^0 and a function $n(\cdot)\in C^1[0, \frac{7}{3}]$ such that (i) $b^0\leq 0$ and $|b_0|+|n(\frac{7}{3})|>0$, (ii) with $n(\frac{7}{3})=b^1$,

$$n(t)=\begin{cases}\frac{2}{3}b^0+b^1+(b^1-\frac{1}{3}b^0)-\frac{1}{2}b^0(t-\frac{1}{3})^2, & t\in[0, \frac{1}{3}]\\ b^1-b^0(t-1), & t\in[\frac{1}{3}, \frac{4}{3}]\\ b^1, & t\in[\frac{4}{3}, 1]\end{cases}$$

and

$$\text{(iii) } u^*(t)=\begin{cases}\mathrm{sgn}(b^0+n(t)), & 0\leq t<\frac{4}{3}\\ x^*(t-1), & \frac{4}{3}\leq t\leq\frac{7}{3}\end{cases}$$

Arguing as in Example 2.1, one may easily obtain the following solution to $(\widetilde{P-w})$ with $w=\frac{7}{3}$.

$$x^*(t)=\begin{cases}2, & -1\leq t\leq 0\\ -3t+2, & 0\leq t\leq\frac{5}{12}\\ -t+\frac{7}{6}, & \frac{5}{12}\leq t\leq 1\\ \frac{3}{2}(t-\frac{4}{3})^2, & 1\leq t\leq\frac{4}{3}\\ 0, & \frac{4}{3}\leq t\leq\frac{7}{3}\end{cases}$$

$$u^*(t)=\begin{cases}-1, & 0\leq t<\frac{5}{12}\\ +1, & \frac{5}{12}\leq t<\frac{4}{3}\\ -3(t-1)+2 & \frac{4}{3}\leq t<\frac{17}{12}\\ -(t-1)+\frac{7}{6} & \frac{17}{12}\leq t<2\\ \frac{3}{2}(t-\frac{7}{3})^2 & 2\leq t\leq\frac{7}{3}\end{cases}$$

Since the pair $(x^*(\cdot), u^*(\cdot))$ given above is also a solution of the problem $(P-w)$ with $w=\frac{7}{3}$, it follows that the pair $(x^*(\cdot), u^*(\cdot))$ is a time-optimal solution of the problem with minimum time $t^*=\frac{7}{3}$.

2.5 Generalized Target Condition

The maximum principle derived in Section 2.2 is for the case when all components of the target function are given. Similar results can be obtained for more general target sets in function space including the case when some of the components of the final function are free. This extension is given below. Since modification to include delays in the control variables is straightforward, for the sake of brevity, systems with delays in the state variable only are investigated.

Let $I=[t, t_1]$ and $I_1=[t_1-h, t_1]$. Let $k\leq n$ be a given positive integer and let $g(x, t)=g^1(x,t,\cdots, g^k(x,t))$ be a vector function on $R^n\times I_1$ which is continuously differentiable in t for each x and twice continuously differentiable in x for each t. Further, for each $i=1,2,\cdots,k$, $j=1,2,\cdots,n$, $l=1,2,\cdots,n$, the derivative $\frac{\partial}{\partial t}g^i(x,t)$ is differentiable in x for each $t\varepsilon I_1$ and the second derivatives $\frac{\partial^2}{\partial x^j\partial x^l}g^i(x, t)$ and $\frac{\partial^2}{\partial x^j\partial_t}g^i(x, t)$ are continuous on $R^n\times I_1$. Let Ω and $\phi(\cdot)$ be the same as in Section 2.1.

Consider the problem of finding an optimal control $u(t)\in\Omega$ on I such that the response of the system

$$\dot{x}(t)=f(x(t), x(t-h), u(t), t)$$

with initial function $x(t)=\phi(t)$ on $[t_0-h, t_0]$ satisfies the target condition

$$g^i(x(t), t)=0, \ t\varepsilon I_1, \ i=1,2,\cdots,k,$$

while minimizing the cost functional

$$C(u(\cdot))=\int_{t_0}^{t_1} f^0(x(t), x(t-h), u(t), t)dt.$$

Obviously, if the first k components of the final function ψ are specified and the rest are free, then $g(x, t)$ should be defined by

$$g^i(x,t)=x^i-\psi^i(t),\ i=1,2,\cdots,k.$$

For each (x,y) and $t\varepsilon I_1$, let $\tilde{w}(x,y,t)$ denote the set of all $u\varepsilon\Omega$ such that

(i) $g_x(x,t)f(x,y,u,t)+g_t(x,t)=0$

(ii) $g_x(x,t)f_u(x,y,u,t)[g_x(x,t)f_u(x,y,u,t)]^T$ is nonsingular,

(iii) $\{g_x(x,t)f_u(x,y,u,t)v\,|\,v\varepsilon R^m,\ q_u{}^i(u)v<0$ if $q^i(u)=0,\ i=1,\cdots,r\}=R^k$.

Let $\bar{x}(t)$ be the response of the system for an admissible control $\bar{u}(t)$, and let $u(t)$ be an admissible control. For each $t\varepsilon I_1$, let $G(t, u(t))$ denote the $k\times k$ matrix

$$G(t, u(t))$$
$$=g_x(\bar{x}(t), t)f_u(\bar{x}(t), \bar{x}(t-h), u(t), t)[g_x(\bar{x}(t), t) f_u(\bar{x}(t), \bar{x}(t-h), u(t), t)]^T.$$

Define the $k\times k$ matrix function $\tilde{G}(t, u(t))$ on I_1 by $\tilde{G}(t, u(t))=G(t, u(t))^{-1}$ if $G(t, u(t))$ is nonsingular at t and $\tilde{G}(t, u(t))=0$ otherwise. In the remainder of this section, the definition of a regular control in Sec. 2.2 is replaced by the following definition: An admissible control $u(t)$ is called regular with respect to $\bar{x}(t)$ if

(i) $g_x(\bar{x}(t), t)f(\bar{x}(t), \bar{x}(t-h), u(t), t)+g_t(\bar{x}(t), t)=0$ a.e. on I_1.

(ii) For almost all $t\varepsilon I_1$, $G(t, u(t))$ is nonsingular and each component of the matrix function $G(t, u(t))$ on I_1 is in $L_\infty(I_1)$.

(iii) For any $z\in L_\infty{}^k(I_1)$, there exists a function δu_z in $A_0(u(\cdot))$ such that $z(t)=g_x(\bar{x}(t), t) f_u(\bar{x}(t), \bar{x}(t-h), \delta u(t), t)\delta u_z(t)$ a.e. on I_1.

Define $k\times n$ matrix functions $G_t(x, t)$ and $G_i(x, t)$, $i=1,2,\cdots,n$, by

$$G_t(x, t)=\left|\ \alpha_{jl}=\frac{\partial^2}{\partial x^l\partial t}g^j(x, t)\right|\ j=1,2,\cdots,k,\ l=1,2,\cdots,n,$$

$$G_i(x, t)=\left|\ \alpha_{jl}=\frac{\partial^2}{\partial x^l\partial x^i}g^j(x, t)\right|\ j=1,2,\cdots,k,\ l=1,2,\cdots,n$$

respectively.

Theorem 2.3. NECESSARY CONDITION. Let $u^*(t)$ with response $x^*(t)$ be an optimal control, and let $u^*(t)$ be regular with respect to $x^*(t)$. Then, there exist a vector $b=(b^0, b^1,\cdots,b^k)\in R^{k+1}$, functions $\nu\in L_\infty{}^k(I_1)$, $\mu\in L_\infty{}^r(I_1)$ and an absolutely continuous n-vector function $n(t)$ on I such that

(i) $|\hat{b}|>0,\ b^0\leq 0$

(ii) $-\dot{n}(t)=+b^0[f_x{}^0(x^*(t), x^*(t-h), u^*(t), t) +f_y{}^0(x^*(t+h), x^*(t), u^*(t+h), t+h)] +n(t)f_x(x^*(t), x^*(t-h), u^*(t), t) +n(t+h)f_y(x^*(t+h), x^*(t), u^*(t+h), t+h),$ a.e. on $[t_0,t_1-2h]$;

$-\dot{n}(t)=+b^0[f_y{}^0(x^*(t), x^*(t-h), u^*(t), t) +f_y{}^0(x^*(t+h), x^*(t), u^*(t+h), t+h)] +n(t)f_x(x^*(t), x^*(t-h), u^*(t), t) +n(t+h)f_y(x^*(t+h), x^*(t), u^*(t+h), t+h) +\nu(t+h)g_x(x^*(t+h), t+h)f_y(x^*(t+h),$

$x^*(t),\ u^*(t+h),\ t+h),$

a.e. on $[t_1-2h,\ t_1-h]$;

$-\dot{n}(t)=b^0 f_x^0(x^*(t),\ x^*(t-h),\ u^*(t),\ t)$
$+n(t)f_x(x^*(t),\ x^*(t-h),\ u^*(t),\ t)$
$+\nu(t)\ \sum_{i=1}^{n} f^i(x^*(t),\ x^*(t-h),\ u^*(t),\ t)G_i(x^*(t),t)$
$+g_x(x^*(t),\ t)f_x(x^*(t),\ x^*(t-h),\ u^*(t),t)+G_t(x^*(t),\ t)]$, a.e. on $[t_1-h,\ t_1]$.

(iii) $n(t_1)=bg_x(x^*(t_1),\ t_1)$, where $b=(b^1,b^2,\cdots,b^k)$;

(iva) $b^0 f^0(x^*(t),\ x^*(t-h),\ u^*(t),\ t)+n(t)f(x^*(t),\ x^*(t-h),\ u^*(t),\ t)$
$=\max_{u\varepsilon\Omega}[b^0 f^0(x^*(t),\ x^*(t-h),\ u,\ t)+n(t)f(x^*(t),\ x^*(t-h),\ u,\ t)]$
for almost all $t\varepsilon I/I_1$,

(ivb) $b^0 f^0(x^*(t),\ x^*(t-h),\ u^*(t),\ t)+n(t)f(x^*(t),\ x^*(t-h),\ u^*(t),\ t)$
$=\max_{u\varepsilon\omega(x^*(t),\ x^*(t-h),\ t)}[b^0 f^0(x^*(t),\ x^*(t-h),\ u,\ t)+n(t)f(x^*(t),\ x^*(t-h),\ u,\ t)]$
for almost all $t\varepsilon I_1$,

(v) $b^0 f_u^0(x^*(t),\ x^*(t-h),\ u^*(t),\ t)$
$+n(t)f_u(x^*(t),\ x^*(t-h),\ u^*(t),\ t)$
$+\mu(t)q_u(u^*(t),\ t)$
$+\nu(t)g_x(x^*(t),t)f_u(x^*(t),x^*(t-h),u^*(t),t)$
$=0$, a.e. on I_1

(vi) $\mu^j(t)\leq 0$ a.e. on I_1, $j=1,2,\cdots,r$

(vii) $\mu^j(t)q^j(u^*(t))=0$ a.e. on I_1, $j=1,2,\cdots,r$.

To prove the above necessary condition, again let $\hat{x}=(x^0,\ x)\varepsilon R^{n+1}$, $\hat{y}=(y^0,\ y)\varepsilon R^{n+1}$ and let

$x^i(x)=g^i(x,\ t_1),\ i=1,2,\cdots,k,$
$x^0(\hat{x})=x^0$
$p^i(\hat{x},\hat{y},u,t)=g_x{}^i(x,t)f(x,y,u,t)+g_t{}^i(x,t),$
$i=1,2,\cdots,k.$

Then the problem can be reformulated as follows: Find a bounded measurable control $u(t)$ on I with response $\hat{x}(t)=(x^0(t),\ x(t))$ such that

$\dot{\hat{x}}(t)=\hat{f}(\hat{x}(t),\ \hat{x}(t-h),\ u(t),\ t)$ a.e. on I,
$\hat{x}(t)=(0,\ \phi(t)),\ t\varepsilon I_0,$
$x^i(\hat{x}(t_1))=0,\ i=1,2,\cdots,k,$
$p^i(\hat{x}(t),\ \hat{x}(t-h),\ u(t),\ t)=0$ a.e. on I_1,
$i=1,2,\cdots,k,$
$q^i(u(t))\leq 0$ a.e. on I, $i=1,2,\cdots,r,$

and $x^0(\hat{x}(t_1))$ is minimum. Thus the problem is essentiallz of the same form as that in Section 2.2.

If appropriate restrictions are imposed on the problem data, the above necessary condition (Theorem 2.3) for generalized targets is sufficient for optimality. To be specific, consider the linear system

$$\dot{x}(t)=A_0(t)x(t)+A_1(t)x(t-h)+B(t)u(t),$$

where the coefficient matrix functions are continuous on $[t_0,\ t_1]$. The convex control set Ω is given by

$$\Omega=\{u\,|\,u\varepsilon R^m,\ q^i(u)\leq 0,\ i=1,2,\cdots,r\}$$

where $q^i(u)$ is convex and continuously differentiable in u for each i. The cost functional $C(u(\cdot))$ is given by

$$C(u(\cdot))=\int_{t_0}^{t_1}[s^0(x(t),\ x(t-h),\ t)+c^0(u(t),\ t)]dt$$

where $s^0(x,y,t)$ is continuously differentiable and convex in x and y, for each t, and is continuous in for each $(x,\ y)$. Similarly, $c^0(u,\ t)$ is continuously differentiable and convex in u, for each t, and is continuous in t for each u. The final function condition is given by

$$N(t)x(t)+d(t)=0,\ t\varepsilon I_1,$$

where $N(t)$ is a continuously differentiable $(k\times n)$-matrix function on I_1, and $d(t)$ is a k-vector continuously differentiable function on I_1.

Let

$f(x,y,u,t)=A_0(t)x+A_1(t)y+B(t)u$
$f^0(x,y,u,t)=s^0(x,y,t)+c^0(u,t)$
$g(x,t)=N(t)x+d(t).$

Theorem 2.4. SUFFICIENT CONDITION. Let $u^*(t)$ be an admissible control with response $x^*(t)$. Suppose $x^*(t)=\phi(t)$ on I_0 and $g(x^*(t),\ t)=0$ on I_1. Further, suppose there exist a vector $\hat{b}=(b^0,\ b)\in R^{k+1}$ with $b^0<0$, functions $\nu\in L_\infty{}^k(I_1)$, $\mu\in L_\infty{}^r(I_1)$ and an absolutely continuous function $n(t)$ on I such that the conditions (ii)-(vii) in Section 5, except (ivb), are satisfied. Then $u^*(t)$ is an optimal control

Example 3. Consider a second-order delay system with a scalar input

$$\dot{x}^1(t)=x^2(t-1)+u(t),\qquad 0\leq t\leq 2$$
$$\dot{x}^2(t)=x^1(t),\qquad 0\leq t\leq 2$$

Initially, $x^1(t)=x^2(t)=0$ on $[-1,\ 0]$. The problem is to find a measurable control $u(t)$ on $[0,\ 2]$ such that the first component of the system response is driven to

$x^1(t)=1, \qquad 1\leq t\leq 2$

while minimizing the quadratic cost functional

$$C(u(\cdot))=\int_0^2\frac{1}{2}[u(t)]^2dt.$$

Let $u^*(t)$ with the response $x^*(t)=(x^{*1}(t),\ x^{*2}(t))$ be an optimal control, and suppose it is regular with respect to $x^*(t)$. By Theorem 2.3, there exist a vector $b=(b^0,\ b^1)\in R^2$, functions $\nu(\cdot)\in L_\infty[1,\ 2]$ and $n(t)=(n^1(t),\ n^2(t))$ on [0, 2] such that the conditions (i) through (v) with $\mu=0$ are satisfied. By the regularity assumption, $u^*(t)=-x^{*2}(t-1)$ on [1, 2]. Setting $b^0=-1$, the condition (v) reduces to

$$n^1(t)+\nu(t)=u^*(t) \qquad \text{a.e. on } [1,\ 2].$$

Therefore, solving the adjoint equations (ii)~(iii), one can obtain

$$n^1(t)=\begin{cases} b^1 & \text{on } [1,\ 2] \\ b^1+\int_t^1\int_s^1 x^{*2}(\tau)d\tau\ ds & \text{on } [0,\ 1] \end{cases}$$

$$n^2(t)=\begin{cases} 0 & \text{on } [1,\ 2] \\ \int_t^1 x^{*2}(t)dt & \text{on } [0,\ 1] \end{cases}$$

The maximum condition (iva) implies

$$u^*(t)=n^1(t)$$

Using the boundary conditions $x^1(0)=x^2(0)=0$ and $x^1(1)=1$, one can solve the system equation to obtain the following solution.

$$x^{*1}(t)=\begin{cases} A_1e^t+A_2e^{-t}+B_1\cos t+B_2\sin t & \text{on}[0,\ 1] \\ 1 & \text{on}[1,\ 2] \end{cases}$$

$$x^{*2}(t)=\begin{cases} A_1e^t-A_2e^{-t}+B_1\sin t-B_2\cos t & \text{on}[0,\ 1] \\ (t-1)+x^{*2}(1) & \text{on}[1,\ 2] \end{cases}$$

$$u^*(t)=\begin{cases} A_1e^t-A_2e^{-t}-B_1\sin t+B_2\cos t & \text{on}[0,\ 1] \\ -x^{*2}(t-1) & \text{on}[1,\ 2] \end{cases}$$

where

$$\Delta=(\sin 1)\cosh 1+(\cos 1)\sinh 1$$

$$A_1=\frac{1}{4\Delta}\{(\sin 1-\sinh 1)-(\cos 1+\cosh 1)\}$$

$$A_2=\frac{1}{4\Delta}\{(\sin 1-\sinh 1)-(\cos 1+\cosh 1)\}$$

$$B_1=\frac{-1}{2\Delta}(\sin 1-\sinh 1)$$

$$B_2=\frac{1}{2\Delta}(\cos 1+\cosh 1)$$

3. CONCLUDING REMARK

In this paper, the problem of optimally controlling time-delay systems to function target was studied. By applying the abstract mathematical programming technique of Neustadt for a reformulated problem is function space, a maximum principle of Pontryagin-type was derived and further, it was also shown that the necessary condition is almost sufficient for optimality. To show the effectiveness of the result, several example problems were solved.

The results in the paper can be easily extended to the case when there are multiple delays in the system state and/or control variables. A certain class of bounded state space problems may handled in a similar manner as in [11], and will be discussed in a future work.

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