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The Stress Distribution in a Long Circular Cylinder under a Discontinuous Boundary Conditions on the Curved Surface

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曲面上에 不連續境界條件을 갖는 圓柱의 應力分布

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抄 錄

이논문에서는 표면의 일부에 旣知의 應力을 받는 긴 圓柱內의 應力分布를 구하는 問題를 고찰하였다. 문제를 혼합경계치조건 에서 발생되는 쌍적분방정식의 解를 구하는 문제로 간단히한 後에 제 2종 Fredholm積分方程式을 해결하는 문제로 하였다. 이 積分方程式의 數値解를 전자계산기에 의하여 구한 다음 圖示하였다.

I. Introduction.

Mixed boundary value problems concerning with long circular cylindrical geometry have been investigated by Srivastav and Lee [6] and others [1, 2, 4] and much of them discuss the cases when the axisymmetric boundary conditions of mixed type are prescribed on the plane which is perpendicular to the curved surface of the cylinder. However, Tranter and Craggs [5] have discussed the problem involving long circular cylinder when a discontinuous pressure is applied to the curved surface. Number of years ago, Vaughan and Allwood [7] obtained dual series equations in course of studying the constriction of an elastic cylinder under an axial compression, and they solved it approximately.

In this paper, the problem of determining the stress distribution in a long circular cylinder, when the mixed boundary conditions are given on the curved surface of it, is considered.

We take the axis of a half infinitely long circular cylinder to be z-axis. We employ cylindrical polar coordinates (ρ, θ, z), and take for convenience the radius of cylinder to be our unit of length. We employ Love's notation throughout this note.

When the radial component of the displacement vector vanishes on the portion of the surface $0 \leq z \leq c$, and the rest of the surface is subject to a known pressure, our problem of determing the stress distribution in a long circular cylinder is equivalent to that of solving biharmonic equation when the curved surface $\rho=1$ is subjected to the conditions

$$\sigma_{\rho z}=0, \quad 0 \leq z < \infty \tag{1.1}$$

$$u_{\rho}=0, \quad 0 \leq z < a \tag{1.2}$$

$$\sigma_{\rho\rho}=-2\mu f(z), \quad a \leq z < \infty \tag{1.3}$$

where μ is the rigidity modulus. The fact that the plane boundary $z=0$ is stress-free implies that on the surface $z=0$, we have

$$\sigma_{\rho z}=0, \tag{1.4}$$

$$\sigma_{zz}=0, \tag{1.5}$$

II. Derivation of the Fredholm Integral Equation of the Second Kind.

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It is well-known(cf.[5]) that the displacement and stress components in the axially symmetric case is given by the following equations

$$u_\rho=-\frac{1}{2\mu}\cdot\frac{\partial^2\chi}{\partial_\rho\partial z},\quad \sigma_{\rho\rho}=\frac{\partial}{\partial z}\left(\eta\nabla^2\chi-\frac{\partial^2\chi}{\partial_\rho^2}\right)$$

$$\sigma_{\rho z}=\frac{\partial}{\partial_\rho}\left[(1-\eta)\nabla^2\chi-\frac{\partial^2\chi}{\partial z^2}\right],$$

$$\sigma_{zz}=\frac{\partial}{\partial z}\left[(2-\eta)\nabla^2\chi-\frac{\partial^2\chi}{\partial z^2}\right] \tag{2.1}$$

where $\chi(\rho, z)$ is an axisymmetric function and η is Poisson's ratio.

A suitable type of biharmonic function for a problem of this type is defined by the equation

$$\chi=2\mu\int_0^\infty B(\xi)\left[-\left\{\frac{I_0(\xi)}{I_1(\xi)}+\frac{2(1-\eta)}{\xi}\right\}I_0(\xi\rho)+\rho I_1(\xi\rho)\right]\cos\xi z d\xi-2\mu\sum_{n=1}^\infty\lambda_n^{-3}F_n(2\eta-1+\lambda_n z)e^{-\lambda_n z}J_0(\lambda_n\rho) \tag{2.2}$$

where λ_n is the root of the equation $J_1(\lambda_n)=0$. A solution of this form automatidally satisfies equations (1.1) and (1.5). The corresponding expressions for $\sigma_{\rho\rho}$ and u_ρ on the curved surface $\rho=1$ are given by the equations

$$\sigma_{\rho\rho}=-2\mu\int_0^\infty B(\xi)I_1(\xi)\xi^2\left[\xi\left(\frac{I^2_0(\xi)}{I^2_1(\xi)}-1\right)-\frac{2(1-\eta)}{\xi}\right]\sin\xi z d\xi-2\mu\sum_{n=1}^\infty F_nJ_0(\lambda_n)(2-\lambda_n z)e^{-\lambda_n z}$$

$$u_\rho=-2(1-\eta)\int_0^\infty B(\xi)I_1(\xi)\xi\sin\xi z d\xi$$

Thus it follows that the boundary conditions (1.2) and (1.3) are satisfied if $B(\xi)$ and F_n satisfy the dual integral equations

$$\int_0^\infty B(\xi)I_1(\xi)\xi^2\left[\xi\left(\frac{I_0^2(\xi)}{I^2_1(\xi)}-1\right)-\frac{2(1-\eta)}{\xi}\right]\sin\xi z d\xi+\sum_{n=1}^\infty F_nJ_0(\lambda_n)(2-\lambda_n z)e^{-\lambda_n z}=f(z),\quad a\leq z<\infty \tag{2.3}$$

$$\int_0^\infty B(\xi)I_1(\xi)\xi\sin\xi z d\xi=0,\quad 0\leq z<a \tag{2.4}$$

It is known [3, Vol. 1] that the equation(2.4) is automatically satisfied if $B(\xi)$ is written in terms of an unknown function $g(t)$ through the equation

$$B(\xi)=\frac{1}{I_1(\xi)\xi}\int_a^\infty tg(t)J_0(\xi t)dt \tag{2.5}$$

We decompose the equation (2.3) in the following way

$$\int_0^\infty B(\xi)I_1(\xi)\xi^2\{1-k(\xi)\}\sin\xi z d\xi+\sum_{n=1}^\infty F_nJ_0(\lambda_n)(2-\lambda_n z)e^{-\lambda_n z}=f(z),\quad a\leq z<\infty \tag{2.6}$$

where,

$$k(\xi)=\left[1-\xi\left(\frac{I^2_0(\xi)}{I^2_1(\xi)}-1\right)+\frac{2(1-\eta)}{\xi}\right]$$

Thus, if we substitute the value of $B(\xi)$ from the equation (2.5) into the equation (2.6), we obtain following Abel type integral equation

$$-\frac{\partial}{\partial z}\int_z^\infty\frac{tg(t)dt}{\sqrt{t^2-z^2}}-\int_a^\infty ug(u)\int_0^\infty\xi k(\xi)J_0(\xi u)\sin\xi z d\xi du+\sum_{n=1}^\infty F_nJ_0(\lambda_n)(2-\lambda_n z)e^{-\lambda_n z}=f(z)$$

If we invert this Abel type integral equation, making use of the known relations [3, Vol. I]

$$\int_t^\infty\frac{\sin\xi z dz}{\sqrt{z^2-t^2}}=\frac{\pi}{2}J_0(\xi t),$$

$$\int_t^\infty\frac{ze^{-\lambda_n z}dz}{\sqrt{z^2-t^2}}=K_0(\lambda_n t),$$

and

$$\int_t^\infty\frac{ze^{-\lambda_n z}dz}{\sqrt{z^2-t^2}}=tK_1(\lambda_n t)$$

we obtain the following Fredholm integral equation of the second kind

$$g(t)-\int_a^\infty ug(u)\int_0^\infty\xi k(\xi)J_0(\xi u)J_0(\xi u)d\xi du+\frac{4}{\pi}\sum_{n=1}^\infty F_nJ_0(\lambda_n)K_0(\lambda_n t)-\frac{\lambda_n t}{2}K_1(\lambda_n t)\}=P(t) \tag{2.7}$$

where

$$P(t)=\frac{2}{\pi}\int_t^\infty\frac{f(z)dz}{\sqrt{z^2-t^2}}$$

Complete solution is obtained by satisfying the boundary condition (1.5). It can be easily shown that $\sigma_{\rho z}$ on the surface $z=0$ is given by the equation

$$\sigma_{\rho z}=2\mu\int_0^\infty\xi^3B(\xi)\left[-\frac{I_0(\xi)}{I_1(\xi)}I_0(\xi\rho)+\rho I_1(\xi\rho)\right]d\xi-2\mu\sum_{n=1}^\infty F_nJ_1(\lambda_n\rho)$$

From the condition (1.5), we get

$$\sum_{n=1}^\infty F_nJ_1(\lambda_n\rho)=\int_0^\infty B(\xi)\xi^3\left[-\frac{I_0(\xi)}{I_1(\xi)}I_1(\xi\rho)+\rho I_0(\xi\rho)\right]d\xi$$

From the Fourier-Bessel series,we obtain the equation

$$\frac{1}{2}F_nJ_0^2(\lambda_n)=-\int_0^\infty B(\xi)\xi^3\left[\frac{I_0(\xi)}{I_1(\xi)}\int_0^1\rho I_1(\xi\rho)J_1(\rho\lambda_n)d\rho-\int_0^1\rho^2 I_0(\xi\rho)J_1(\lambda_n\rho)d\rho\right]d\xi \tag{2.8}$$

If we denote

$$H(\xi,\lambda_n)=\int_0^1\rho I_1(\xi\rho)J_1(\lambda_n\rho)d\rho,$$

it can be easily shown that [8]

$$H(\xi,\lambda)=-\frac{\lambda}{\xi^2+\lambda^2}J_0(\lambda)I_1(\xi) \tag{2.9}$$

and

$$\int_0^1\rho^2 I_0(\xi\rho)J_1(\lambda\rho)d\rho=\frac{1}{\xi}\frac{\partial}{\partial\xi}\xi H(\xi,\lambda)=-\lambda\frac{J_0(\lambda)}{\xi^2+\lambda^2}\left[I_0(\xi)-\frac{2\xi}{\xi^2+\lambda^2}I_1(\xi)\right] \tag{2.10}$$

Putting the equations (2.9) and (2.10) into the equation (2.8), we obtain, after simplification

$$F_nJ_0(\lambda_n)=4\lambda_n\int_0^\infty\xi^4\frac{B(\xi)I_1(\xi)}{(\xi^2+\lambda_n^2)^2}d\xi \tag{2.11}$$

Substitution of the value of $B(\xi)$ from the equation (2.5) into the equation (2.11) yields

$$F_nJ_0(\lambda)=4\lambda\int_a^\infty tg(t)F(t,\lambda)dt$$

where

$$F(t,\lambda)=\int_0^\infty\frac{\zeta^3J_0(\zeta t)}{(\lambda^2+\zeta^2)^2}d\zeta$$

Further

$$F(t,\lambda)=\Psi(t,\lambda)+\frac{\lambda}{2}\frac{\partial\Psi(t,\lambda)}{\partial\lambda}$$

where

$$\Psi(t,\lambda)=\int_0^\infty\frac{\zeta J_0(\zeta t)}{\lambda^2+\zeta^2}d\zeta$$

We find that, from [4, Vol. 2]

$$\Psi(t,\lambda)=K_0(\lambda t)$$

Hence, we have

$$F(t,\lambda)=4\{K_0(\lambda t)-\frac{\lambda t}{2}K_1(\lambda t)\}$$

and, thus,

$$F_nJ_0(\lambda_n)=4\lambda_n\int_a^\infty g(u)u[K_0(\lambda_n u)-\frac{\lambda_n u}{2}K_1(\lambda_n u)]du$$

Upon inserting the above expression into the equation (2.7), we finally obtain following Fredholm integral equation of the second kind.

$$g(t)-\int_a^\infty g(u)K(u,t)du=P(t) \tag{2.12}$$

where

$$K(u,t)=u\left[\int_0^\infty\xi k(\xi)J_0(\xi u)J_0(\xi t)d\xi-\frac{16}{\pi}\sum_{n=1}^\infty\lambda_n h(\lambda_n u)h(\lambda_n t)\right]$$

with

$$h(\lambda_n t)=K_0(\lambda_n t)-\frac{\lambda_n t}{2}K_1(\lambda_n t)$$

III. Numerical Solution of the Integral Equation.

In this section numerical solution of the integral equation (2.12) is considered when $f(z)=\frac{1}{z}$. Thus the governing integral equation is

$$g(t)-\int_a^\infty g(u)K(u,t)du=\frac{1}{t}$$

The integral equation was reduced to the set of simultaneous linear equations by replacing the integral by finite sum using 7-point Laguerre quadrature. The integral occuring in the kernel was also calculated by using the same quadrature formula. To secure the convergence of the later integral we have used asymtotic expansion

$$\frac{I^2_0(\xi)}{I^2_1(\xi)}\doteqdot 1+\frac{1}{\xi}+\frac{1}{\xi^2}+\frac{1}{8\xi^3},\qquad \xi>\pi$$

The result of the calculations of $g(t)$ is shown graphically in Fig. 1 for $a=2$ and $\eta=0.3$.

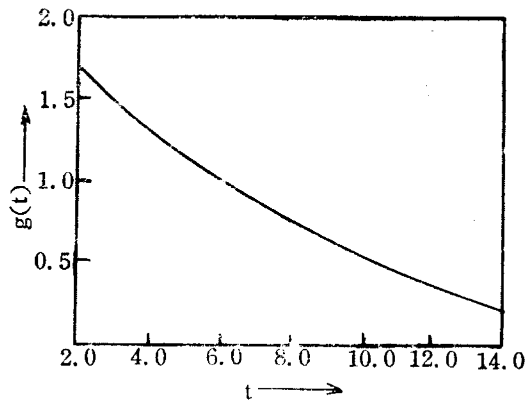

Fig. 1

References

1. Collins, W.D., *Proc. Edin. Math*, Soc.(1962), 13 (69)
2. Das, B.R., *Int. J. Engng. Sci.*, 1968, Vol. 6, pp. 497-516.

3. Erdelyi, A., *Tables of Integral Transforms,* Vol. I & II, McGraw-Hill (1954).
4. Sneddon, I.N., and Tait, R.J., *Int. J. Engng. Sci.,* 1966, Vol. I, pp. 391-409.
5. Sneddon, I.N., *Fourier Transform,* McGraw-Hill (1951).
6. Srivastav, R.P. and Lee, D., *Int. J. Engng. Sci.,* 1972, Vol. 10, pp. 217-232.
7. Vaughan, H. and Allwood, D., *Proc. Camb. Phil. Soc.,* 1972(499).
8. Watson, G.N., *A Treatise on the Theory of Bessel Functions,* 2nd Edition, Cambridge Univ.Press.