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Optimal Design of a Straight Fin by a Generalized Steepest Descent Method

by

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일반적인 최적설계방법에 의한 최적 냉각휜의 설계

곽 병 만

초 록

냉각용 Fin의 설계문제를 일반적인 최적설계문제로 바꾸어서 일반화된 Steepest Descent 방법에 의한 수치적 방법을 도입하여 해결하였다.

보다 실제적인 문제를 다룰 수 있도록 여러 가지 제한조건을 고려한 Fin의 최적곡선 모양의 해를 얻었으며 이 방법의 유용성을 보였다. 사다리꼴의 Fin 설계예에서 위 방법을 이용한 해와 직접 계산에 의한 열전달량의 등고선 그람으로부터 구한 해와 일치함을 보였다.

Introduction

The design of optimal cooling fins has been of continued interest for several decades (1, 2, 3). The design problem as formulated in the literature however does not allow enough flexibility for general type of constraints, and also the methods of solution are not general enough to handle slightly varied problems. In Ref. 3, the shape of fin obtained is of wavy form and the thickness to height ratio is too small to have any meaning.

The problem as treated in this paper is to select an optimum geometry of a cooling fin for the maximum amount of convective heat transfer. It is formulated as a general optimal design problem, so that various physical constraints can be imposed.

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Statement of the Problem

The basic problem is to dissipate the maximum amount of heat, using a fin of given height, where the base temperature T_b and surrounding temperature T_∞ are both given and are assumed constant. The geometry of the fin to be designed is shown in Fig. 1. Steady state one dimensional conduction is assumed. Heat transfer properties are also assumed constant.

The governing equations for the temperature distribution in the fin can be obtained by considering heat balance in a differential element and Fourier's law of heat conduction as follows:

$$\frac{d}{d_x}\left(-kA_x\frac{dT}{dx}\right)+\frac{2h}{\cos\varphi}(T-T_\infty)=\dot{q}A_x, \quad (1)$$

where k is the thermal conductivity and h is heat transfer coefficient. Since the fin usually does not generate heat therefore $\dot{q}=0$.

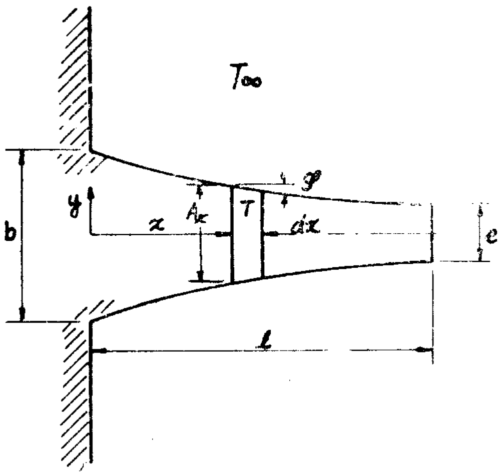

Fig. 1. Geometry of a Straight Fin

The boundary conditions are

$$T\Big|_0 = T_b,\ \frac{dT}{dx}\Big|_l = 0,$$

if the fin has an insulated end, or

$$T\Big|_0 = T_b,\ -k\frac{dT}{dx}\Big|_l = h\,(T-T_\infty)\Big|_l,$$

if the fin has a convecting end.

The objective function chosen is to maximize the heat transferred through the fin; which is the same as to minimize the negative of heat transferred

$$J' = bk\frac{dT}{dx}\Big|_0 \tag{2}$$

Introducing the following dimensionless variables

$$\theta = \frac{T-T_\infty}{T_b-T_\infty},\ t=\frac{x}{l},\ b_1=\frac{b}{l}\ \text{and}\ v=\frac{A_x}{l}\ , \tag{3}$$

Eq. (1) becomes

$$\frac{d}{dt}\left(-v\frac{d\theta}{dt}\right)+\frac{2hl}{k\cos\varphi}\theta=0, \tag{4}$$

with boundary conditions of either

$$\theta(0)=1,\ \frac{d\theta}{dt}(1)=0;\ \text{(insulated end)} \tag{5}$$

or

$$\theta(0)=1,\ \frac{d\theta}{dt}(1)=-\frac{hl}{k}\theta(1);\ \text{(convecting end)} \tag{6}$$

and the function to be minimized is

$$J' = b_1 k\,(T_b-T_\infty)\frac{d\theta}{dt}\Big|_0,$$

or in nondimensional form

$$J=\frac{J'}{k(T_b-T_\infty)}=b_1\frac{d\theta}{dt}\Big|_0 \tag{7}$$

Constraints of practical interest will be imposed in the numerical examples following.

Formulation of Optimal Design Problem and Numerical Technique

The problem described above is a typical case of the general optimal design problem and is formulated as follows. Given a system, the behavior is described by a set of equations called state equations. The behavior is denoted by a set of variables called state variable vector, z. The parameters of the system which the designer wants to choose are included in a design parameter vector, b. If they are functions of time or space variable, they will be called design variables, u. A performance index function is a mathematical representation of the criterion by which the design variables are decided. The general optimal design formulation then can be stated as: To minimize the functional

$$J=\Psi_0(u,\ b,\ z); \tag{8}$$

subject to the system equations

$$K(u,\ b,\ z)=0; \tag{9}$$

where K is a differential or algebraic operator; subject to inequality and/or equality functional constraints,

$$\Psi_i(u,\ b,\ z)\begin{cases}\le 0\\ =0,\ i=1,\cdots,p\end{cases} \tag{10}$$

and subject to inequality or equality design variable constraints,

$$\phi_j(u,\ x)\begin{cases}\le 0\\ =0,\ j=1,\cdots q.\end{cases} \tag{11}$$

A generalized steepest descent method(4, 5, 6) will be employed. The basic philosophy of the method may be summarized as follows. An initial estimate of the design parameters are made. Corresponding to this design, gradients of the functionals and function, constrained in the design space, are obtained. The procedure for obtaining the constrained gradient will be shown for the objective functional, assuming that only design variable u exists. The treatment with both u and b is straightforward (4). A linearized form of the functional is

$$\delta J=\langle \Psi_{0},{}^{T}{}_{u},\ \delta u\rangle+\langle \Psi_{0},{}_{z}{}^{T},\ \delta z\rangle,$$

where a subscript after comma denotes partial derivatives, superscript T denotes transpose and an inner product natation $\langle .,.\rangle$ is used. The key step in eliminating the dependency on δz, is the concept of adjoint equation. The linearized state equation is

$$K_z\delta z+K_u\delta u=0,$$

where K_z and K_u are now linear operators. By defining formal adjoint operators $K_z{}^*$ and $K_u{}^*$ to K_z and K_u, let λ^0 be the solution of

$$K_z{}^*\lambda^0=-\Psi_{0},{}_{z}{}^{T}, \tag{12}$$

with adjoint boundary conditions (4). Then,

$$\begin{aligned}\langle \Psi_{o},{}_{z}{}^{T},\ \delta_z\rangle&=\langle -K_z{}^*\lambda^0,\ \delta z\rangle=\langle -\lambda^0,\ K_z\delta z\rangle\\&=\langle \lambda^0,\ K_u\delta u\rangle=\langle K_u{}^*\lambda^0,\ \delta u\rangle.\end{aligned}$$

Therefore,

$$\delta J=\langle \Psi_{o},{}_{u}{}^{T}+K_u{}^*\lambda^0,\ \delta u\rangle,$$

and the constrained gradient will be

$$\Lambda^J=\Psi_{o},{}_{u}{}^{T}+K_u{}^*\lambda^0. \tag{13}$$

The same procedure applied on the other functionals Eq. (10) which contain state variables allows a linearized formulation of the original optimal design problem in design space. Also, the amount of design change is limited such that the linearization retains its accuracy. The resulting problem is then solved to obtain a desirable design change δu. For details of the method, refer to Ref. 4 and 5, where various numerical examples are also given. A program has been developed and described in Ref. (6), and is used to solve the numerical examples in the following.

Optimal Trapezoidal Fins

As a first numerical example, a symmetric trapezoidal fin is designed for an insulated end. The base width b and tip width e become the design parameters. From the fin geometry,

$$A_x=b-\frac{b-e}{l}x$$

and

$$\cos\varphi=2\left(\left(\frac{b-e}{l}\right)^2+4\right)^{-1/2}.$$

In terms of the dimensionless variables, Eq. (4) becomes

$$\frac{d}{dt}\left\{-(b_1-(b_1-b_2)\,t)\frac{d\theta}{dt}\right\}+\frac{2hl}{k\cos\varphi}\theta=0, \tag{14}$$

where

$$b_2=\frac{e}{l}.$$

To get a first order form of state equations, define

$$z_1=\theta$$

$$z_2=-(b_1-(b_1-b_2)t)\frac{d\theta}{dt}. \tag{15}$$

In this notation, Eq. (14) becomes

$$\left.\begin{aligned}&\frac{dz_1}{dt}=-\frac{z_2}{b_1-(b_1-b_2)t}\\&\frac{dz_2}{dt}=-\frac{2hl}{k\cos\varphi}z_1,\\&\text{with}\\&z_1(0)=1,\ \ z_2(1)=0,\end{aligned}\right\} \tag{16}$$

and Eq. (7) becomes

$$J=-z_2(0). \tag{17}$$

The following design variable constants will be imposed:

$$\begin{aligned}&\Psi_1\equiv b_1-b_{\max}\leq 0,\\&\Psi_2\equiv b_2-b_1\leq 0,\end{aligned} \tag{18}$$

where $b_{\max}$ is a maximum allowable thickness.

The formulations of the optimal design problem in the previons section now applies directly. The adjoint quation (12) becomes,

$$\frac{d\lambda_1}{dt}=\frac{2hl}{k\cos\varphi}\lambda_2,\quad \frac{d\lambda_2}{dt}=\frac{1}{b_1-(b_1-b_2)t}\lambda_1. \tag{19}$$

Since the inequality constraints in Eq. (18) are explicit functions of only the design variables, no adjoint equation with the constraint function need be computed. Solving for λ_1 from the second equation and substituting it into the first equation, one has the second order adjoint differential equation:

$$\frac{d}{dt}\left\{\left(b_1-(b_1-b_2)t\right)\frac{d\lambda_2}{dt}\right\}-\frac{2hl}{k\cos\varphi}\lambda_2=0, \qquad (20)$$

which is the same as Eq. (14).

To determine boundary conditions on λ, consider

$$(\lambda_1\delta z_1+\lambda_2\delta z_2)\Big|_0^1=-\delta z_2(0), \qquad (21)$$

which must hold for all δ_z satisfying a linearized form of the boundary conditions appearing in Eq. (16). In order that Eq. (21) hold identically for all variations δ_z (0) and $\delta_z(1)$ satisfying the linearized boundary conditions of Eq. (16), it is necessary that the following conditions on λ be satisfied.

$$\left.\begin{array}{l}\lambda_1(1)=0\\ \lambda_2(0)=1.\end{array}\right\} \qquad (22)$$

From the second equation in (19), one may replace the condition in Eq. (22) on $\lambda_1(1)$ by a condition on the derivative of $\lambda_2(1)$. In this manner, one obtains the following boundary conditions on λ_2:

$$\left.\begin{array}{l}\frac{d\lambda_2}{dt}(1)=0\\ \lambda_2(0)=1.\end{array}\right\} \qquad (23)$$

Note that the adjoint differential equation (20) with the boundary condition (23) is exactly the same as the differential equation (14). Thus, the solution of the adjoint differential equation is trivially constructed, once the temperature equation is solved. One is now ready to follow the algorithm in Ref. 5.

The constant (Biot number) for heat transfer properties are assumed as

$$\frac{hl}{k}=0.0061,$$

which approximately corresponds to an iron fin exposed to air.

An analytical solution for the temperature distribution is easily obtained as shown in Appendix, but a numerical differential equation solver was used to solve the boundary value problem of Eq. (14). A shooting technique described in (4) gave a solution after 2 iterations, since the equations are linear.

Solutions corresponding to three constraint sets are shown in Table 1. From the initial design estimate of $b_1=0.1$ and $b_2=0.05$ ($J=-0.01159$), the solutions are obtained in less than 15 iterations. The optimal shapes obtained are shown in Fig. 2 with a schematic contour map of the heat transfer function in the design space. The contour is obtained after plotting the heat transfer functions calculated from the formula given in Appendix. The solution obtained numerically is consistent with those from the contour map. From this figure, it is seen that the design variable constraints are of utmost importance for the present problem. If the base width is less than about 25% of the fin length (i.e., $b_1<0.25$), a rectangular fin ($b_1=b_2$) is the solution and if $b_1>0.25$, a triangular fin ($b_2=0$) is the solution. Around the 25% range, a trapezoidal fin is the solution.

Table 1. Solution of Trapezoidal Fin Design

Constraints	$b_2\leq b_1$		
	$b_1\leq 0.2$	$b_1\leq 0.25$	$b_1\leq 0.4$
Solution	$b_1=b_2=0.2$	$b_1=0.25$ $b_2=0.026$	$b_1=0.4$ $b_2=0.0$
Heat Transfer	-0.01198	-0.01193	-0.01217
Shape of Fin	Rectangular	Trapezoidal	Triangular

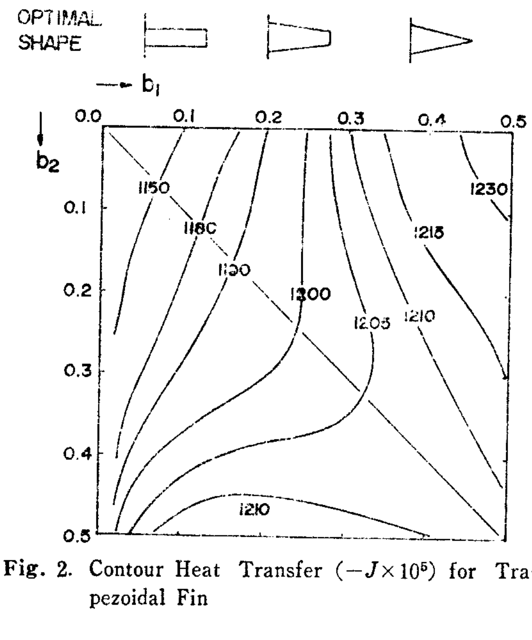

Fig. 2. Contour Heat Transfer ($-J\times 10^5$) for Trapezoidal Fin

Optimal Fin Profiles for the Maximum Heat Dissipation

As a second numerical example, consider the case in which the width, $v(t)$ of fin is chosen as a design variable. Then,

$$\frac{1}{\cos\varphi}=(1+(v')^2/4)^{1/2}. \tag{24}$$

Since the governing Eq. (4) upon substitution of Eq. (24) contains the derivative of v, the problem does not fit the standard optimal design formulation. To circumvent this situation, let

$$v=z_3+b_1, \tag{25}$$

Such that

$$z_3'=v'\equiv u, \tag{26}$$

where

$$v(0)=b_1 \text{ or } z_3(0)=0.$$

The governing equations become

$$\left.\begin{aligned}\frac{dz_1}{dt}&=-\frac{z_2}{z_3+b_1}\\ \frac{dz_2}{dt}&=-\frac{2hl}{k}\sqrt{1+\frac{u^2}{4}}z_1\\ \frac{dz_3}{dt}&=u.\end{aligned}\right\} \tag{27}$$

Boundary conditions corresponding to Eqs. (5) and (6) are: (1) Insulated end,

$$z_1(0)=1,\ z_2(1)=0,\ \text{and}\ z_3(0)=0; \tag{28}$$

or (2) convecting end,

$$z_1(0)=1,\ z_2(1)-\frac{hl}{k}(z_3(1)+b_1)z_1(1)=0,$$

$$\text{and } z_3(0)=0. \tag{29}$$

As before, the heat transfer function is

$$J=-z_2(0) \tag{30}$$

The following constraints are imposed:

$$\Psi_1\equiv b_1-b_{max}\leq 0, \tag{31}$$

and

$$\phi_1\equiv u\leq 0. \tag{32}$$

Equation (32) states that the slope in negative, i.e., the width $v(t)$ is a nonincreasing function of t. One may want to impose $v\geq b_{min}$. This can be expressed, using Eq. (25) as

$$\Psi_2\equiv -z_3(1)-b_1+b_{min}\leq 0, \tag{33}$$

where b_{min} denotes a lower bound of the thickness.

The slope $u(t)$ and the width b_1 are chosen as design variables. Notice that the system is nonlinear and by a simple translation, the appearance of the design parameter b_1 in the initial condition has been eliminated.

Another constraint of practical importance is that the volume of fin be restricted to be less than some given quantity, V. That is,

$$\int_0^1 v(t)dt\leq V,$$

or using Eq. (25),

$$\Psi_3\equiv b_1+\int_0^1 z_3 dt-V\leq 0. \tag{34}$$

Thus, the problem formulated can be stated as selecting the design parameter b_1 and the design variable u to minimize Eq. (30), subject to functional constraints Eqs. (31), (33) and (34), and pointwise constraint Eq. (32).

For the constraint Ψ_3 of Eq. (34), it need not be necessary to set up an associated adjoint system: Using Eq. (27), and integrating by parts,

$$\delta\Psi_3=\delta b_1+\int_0^1\delta z_3 dt=\delta b_1+\int_0^1(1-t)\delta u dt. \tag{35}$$

Hence the desired gradient is

$$\Lambda^3=1-t. \tag{36}$$

To obtain the constrained gradients of J and Ψ_2, the adjoint systems are formulated:

$$\left.\begin{aligned}\frac{d\lambda_1}{dt}&=\frac{2hl}{k}\sqrt{1+\frac{u^2}{4}}\lambda_2,\\ \frac{d\lambda_2}{dt}&=\frac{1}{z_3+b_1}\lambda_1,\\ \frac{d\lambda_3}{dt}&=-\frac{z_2}{(z_3+b_1)^2}\lambda_1.\end{aligned}\right\} \tag{37}$$

For the objective functional, which contains a state variable, the adjoint boundary conditions are obtained as: (1) for an insulated end,

$$\lambda_1(1)=0,\ \lambda_2(0)=1, \text{ and } \lambda_3(1)=0; \tag{38}$$

and (2) for a convecting end,

$$\left.\begin{aligned} &\lambda_2(0)=1,\ \lambda_1(1)+(z_3(1)+b_1)\frac{hl}{k}\lambda_2(1)=0, \\ &\text{and} \\ &\lambda_3(1)+\lambda_2(1)\frac{hl}{k}z_1(1)=0. \end{aligned}\right\} \tag{39}$$

Notice that $\lambda_1=-z_2$ and $\lambda_2=z_1$, hence only the third equation of the adjoint system needs to be solved.

For the minimum thickness constraint Eq. (33) the adjoint boundary conditions are:

$$(1)\ \lambda_1(1)=0,\ \lambda_2(0)=0,\ \lambda_3(1)=-1; \tag{40}$$

and (2)

$$\left.\begin{aligned} &\lambda_1(1)+\lambda_2(1)\frac{hl}{k}(z_3(1)+b_1)=0,\ \lambda_2(0)=0, \\ &\text{and} \\ &\lambda_3(1)+\lambda_2(1)\frac{hl}{k}z_2(1)+1=0, \end{aligned}\right\} \tag{41}$$

corresponding to the insulated and convecting end, respectively. It is seen that solutions of the adjoint differential equations for both boundary conditions above, are

$$\left.\begin{aligned} &\lambda_1(t)=0 \\ &\lambda_2(t)=0 \\ &\lambda_3(t)=-1. \end{aligned}\right\} \tag{42}$$

Thus, the steepest descent algorithm can now be applied in a routine manner.

Three sets of constraints are considered for each end condition. The minimum thickness of fin was restricted to be no less than $b_{min}=0.01$. This was necessary not only for manufacture but for the removel of unstability due to a singularity in Eq. (27) when the fin thickness approaches zero. For numerical solution of the differential equations, the interval was divided into 100 sub-intervals, while only 50 intervals were used for integration and specification of the design variable. The shooting technique gave good convergence less than 5 iterations, for an error of 10^{-7} in the boundary conditions. Convergence was obtained after about 15 iterations for the insulated end case; but for the convecting end case convergence was rather poor. The results obtained are summarized in Tables 2 and 3. The shapes of optimum fins are shown in Fig. 3. The amount of heat transfered per unit area, without a fin is $\frac{hl}{k}=0.0061$. Comparing with this number, the amount of heat dissipation per unit area of fin is shown in parenthesis in Tables 2 and 3. The volume constraint of 0.3 was not tight for the design constraint set 3, where the constraint, $b\leq 0.6$ was dominant.

Table 2. Insulated End Case

Constraints 1: $b_1\leq 2.0$, volume≤ 0.3, $b_{min}\leq 0.01$

Solution: $b_1=1.48$

u;	−4.88	−4.74	−4.60	−4.46	−4.31	−4.17	−4.02	−3.87	−3.72	−3.56
	−3.40	−3.24	−3.08	−2.91	−2.74	−2.56	−2.38	−2.19	−2.00	−1.80
	−1.59	−1.38	−1.15	−0.92	−0.68	−0.45	−0.22	−0.06	−0.05	−0.02
	−0.05	−0.04	−0.04	0.	0.	0.	0.	0.	0.	0.

(The rest are zero).

Objective function; $J=-0.0166$ (Heat dissipation per unit area=0.0112)
Remarks; Volume and minimum thickness constraints were tight.

Constraints 2: $b_1\leq 2.0$, volume≤ 0.3, $b_{min}=0.1$

Solution; $b_1=1.17$

u;	−3.63	−3.52	−3.41	−3.30	−3.19	−3.08	−2.97	−2.86	−2.74	−2.63
	−2.51	−2.39	−2.27	−2.14	−2.02	−1.89	−1.76	−1.62	−1.49	−1.34
	−1.20	−1.05	−0.90	−0.75	−0.59	−0.43	−0.26	−0.09	0.	0.

(The rest are zero).

Objective function; $J=-0.015$ (Heat dissipation per unit area=0.0128)
Remarks; Volume and minimum thickness constraints were tight.

Constraints 3: $b_1 \leq 0.6$, volume≤ 0.3, $b_{min}=0.01$

Solution; $b_1=0.6$

u;	−2.46	−2.39	−2.30	−2.22	−2.12	−2.03	−1.93	−1.81	−1.69	−1.56
	−1.43	−1.26	−1.11	−0.93	−0.74	−0.54	−0.32	−0.11	0.	0.

(The rest are zero).

Objective function; $J=-0.013$ (Heat dissipation per unit area$=0.0216$)

Remarks; $b_1 \leq 0.6$ was tight.

Table 3. Convecting End Case

Constraints 1: $b_1 \leq 2.0$, volume≤ 0.3, $b_{min}=0.01$

Solution; $b_1=1.39$

u;	−5.01	−4.84	−4.66	−4.48	−4.30	−4.12	−3.94	−3.75	−3.56	−3.36
	−3.19	−3.00	−2.80	−2.61	−2.40	−2.21	−2.00	−1.79	−1.57	−1.36
	−1.13	−0.90	−0.66	−0.42	−0.17	0.	0.	0.	0.	0.

(The rest are zero).

Objective function; $J=-0.0167$ (Heat dissipation per unit area$=0.012$)

Remarks; Volume constraint was tight.

Constraints 2: $b_1 \leq 2.0$, volume≤ 0.3, $b_{min}=0.1$

Solution; $b_1=1.1$

u;	−4.69	−4.47	−4.24	−4.00	−3.77	−3.52	−3.29	−3.04	−2.79	−2.53
	−2.30	−2.00	−1.73	−1.44	−1.13	−0.79	−0.45	−0.09	0.	0.

(The rest are zero).

Objective function; $J=-0.0155$ (Heat dissipation per unit area$=0.014$)

Remarks; Minimum thickness constraint was tight.

Constraints 3: $b_1 \leq 0.6$, volume≤ 0.3, $b_{min}=0.01$

Solution; $b_1=0.6$

u;	−2.71	−2.55	−2.38	−2.21	−2.03	−1.84	−1.65	−1.44	−1.21	−0.97
	−0.71	−0.43	−0.14	0.	0.	0.	0.	0.	0.	0.

(The rest are zero).

Objective function; $J=-0.0141$ (Heat dissipation per unit area$=0.024$)

Remarks; $b_1 \leq 0.6$ was tight.

Summary and Conclusions

The optimal design of a cooling fin for a maximum dissipation of heat has been formulated as a general optimal design problem. A generalized steepest descent method is used and shows its versatility to treat any type of constraints and heat transfer objective function.

Several numerical solutions are presented. No comparison, however, was possible with existing literature, because of the generality of the problem formulation treated here. But the solutions obtained are more realistic than those in the literature, due to the practical physical contraints imposed. Although example problems are limited to the maximum heat problems, a minimum weight problem can be solved as well without difficulty.

It is concluded that the overall approach taken here can be utilized effectively in the practical problems of fin design under any possible constraints the designer wants to impose.

Appendix

Solution of the Differential Equation (14)

The quation to be solved is Eq. (14) with boundary

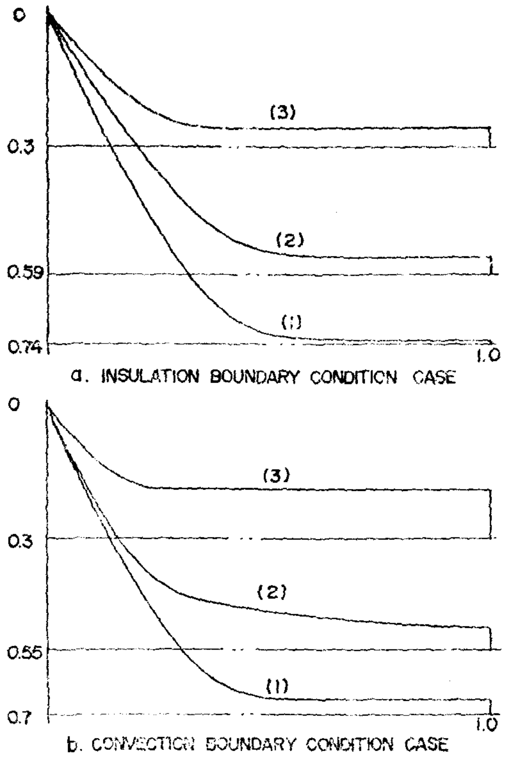

Fig. 3. Optimal Fin Shapes for Maximum Heat Dissipation

conditions

$$\theta(0)=1,\quad \frac{d\theta}{dt}(1)=0.$$

Let

$$b_1-(b_1-b_2)t\equiv u$$

$$\frac{d}{dt}\equiv-(b_1-b_2)\frac{d}{du} \qquad \text{(A-1)}$$

where $b_1 \neq b_2$ is assumed

Then

$$\left.\begin{aligned}&\frac{d^2\theta}{du^2}+\frac{1}{u}\frac{d\theta}{du}-\beta\frac{\theta}{u}=0,\\ &\theta(b_1)=1,\quad \frac{d\theta}{du}(b_2)=0,\end{aligned}\right\} \qquad \text{(A-2)}$$

where

$$\beta=\frac{2hl}{k(b_1-b_2)^2\cos\varphi}. \qquad \text{(A-3)}$$

The general solution of equation (A-2) is

$$\theta=AI_0(2\sqrt{\beta u})+BK_0(2\sqrt{\beta u}), \qquad \text{(A-4)}$$

where I_0 and K_0 are modified Bessel functions.

From the first boundary condition,

$$AI_0(2\sqrt{\beta b_1})+BK_0(2\sqrt{\beta b_1})=1. \qquad \text{(A-5)}$$

Differentiating (A-4),

$$\frac{d\theta}{du}=\{AI_1(2\sqrt{\beta u})-BK_1(2\sqrt{\beta u})\}\sqrt{\frac{\beta}{u}}. \qquad \text{(A-6)}$$

Hence the second boundary condition gives,

$$AI_1(2\sqrt{\beta b_2})-BK_1(2\sqrt{\beta b_2})=0. \qquad \text{(A-7)}$$

Solving for A and B from Eqs., (A-5) and (A-7),

$$\left.\begin{aligned}&A=K_1(2\sqrt{\beta b_2})/\Delta,\\ &B=I_1(2\sqrt{\beta b_2})/\Delta,\end{aligned}\right\} \qquad \text{(A-8)}$$

where

$$\Delta=I_0(2\sqrt{\beta b_1})K_1(2\sqrt{\beta b_2})+I_1(2\sqrt{\beta b_2})K_0(2\sqrt{\beta b_1}). \qquad \text{(A-9)}$$

Now the objective function (7) is given by

$$J=-b_1(b_1-b_2)\frac{d\theta}{du}\Big|_{b_1}$$
$$=\sqrt{\beta b_1}(b_1-b_2)\{AI_1(2\sqrt{\beta b_1})-BK_1(2\sqrt{\beta b_1})\}, \qquad \text{(A-10)}$$

where A and B are given in (A-8), and from (A-3)

$$\beta=\frac{hl}{k}\frac{\sqrt{(b_1-b_2)^2+4}}{(b_1-b_2)^2}. \qquad \text{(A-11)}$$

In case when $b_1=b_2$ Eq. (14) becomes

$$\frac{d^2\theta}{dt^2}-\frac{2hl}{b_1k}\theta=0. \qquad \text{(A-12)}$$

The solution is given as

$$\theta=\frac{e^{\lambda t}}{1+e^{2\lambda}}+\frac{e^{-\lambda t}}{1+e^{-2\lambda}}$$

Where

$$\lambda=\sqrt{\frac{2hl}{b_1k}}$$

Hence

$$J=-b_1\lambda\tanh\lambda. \qquad \text{(A-13)}$$

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