OPPORTUNISTIC REPLACEMENT POLICIES UNDER MARKOVIAN DETERIORATION

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ABSTRACT

Consider a series system of two units, named 1 and 2, respectively. Two units are observed at the beginning of discrete time periods $t=0, 1, 2, \ldots$ and classified as being in one of a countable number of states. Let (i, r) be a state of the system at time t, when the state of unit 1 is i and state of unit 2 is r at time t. Under some conditions, the opportunistic replacement policy that minimizes the expcted total discounted cost or the average cost of maintenance is shown to be characterized by the control limits $i^*(r)$ (a function of r) and $r^*(i)$ (a function of i) : (a) in observed state (i, r), the optimal policy for unit 1 is to replace if $i \geq i^*(r)$ and no action otherwise; (b) in observed state (i, r), the optimal policy for unit 2 is to replace if $r \geq r^*(i)$ and no action otherwise. In addition, this paper also develops optimal policy in the finite time horizon case, where time horizon is fixed or a finite integer valued *r.v.* with known pmf.

1. Introduction

In this paper, we consider the problem of determining the replacement policy for a series sestem of two units. Two units, called unit 1 and unit 2 respectively, are observed at the beginning of discrete time periods $t=0, 1, 2, \ldots\ldots$, and classified as being in one of a countable number of states. After observing the states of both units the observer must make a decision for each unit whether to replace or not. If it costs less to replace two units concurrently than to Jeplace them at different times, the necessary replacement of one unit upon failure may also justify the replacement of the other unit in the system whose failure seems imminent. If the operating costs of the system is higher for a higher level state (more deteriorated) than for a lower level state(less deteriorated), Ross (3) demonstrates the economical *justification of the replacement* before failure, i.e., preventive replacement. The preventive replacement of one unit may also justifies the replacement of the other unit as the replace-

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ment upon failure(corrective replacement) justifies. Suppose the operating cost of the system is a function of the states of both unit 1 and unit 2. Then the decision for the preventive replacement of one unit may depend on the state of the other unit. In other words, if the optimal policy is to replace unit 1 when the states of unit 1 and unit 2 are i and r, it may not be optimal to replace unit 1 when the state of unit 2 is different from r. Thus the optim al decision for unit 1 (unit 2) may be a function of the states of unit 2 (unit 1), that is, the decision for unit 1 (unit 2) is not determined only by the state of unit 1 (unit 2), but also the state of unit 2 (unit 1). The aim of this paper is to charaterize the optimal policy for such an opportunistic replacement model.

One of the few closely related works is that of Sethi (4). He also considered the opportunistic replacement model of two series units. The main difference with our model is that he views the age of each unit as the states of each unit, while in our model, there is no such limitations in the definition of states. Thus our model may be viewed as a kind of generalization of his model. It is found that our optimal replacement policy encompasses his optimal age replacement policy.

In section 2, state space, decision space, cost structure, and transition probabilities of the system are introduced. In addition, we impose some conditions on the costs and transition probabilities, and present preliminary lemma needed for the development of the optimal policy. Section 3 treats the case where time horizon is infinite under the discounted cost criterion. Section 4 treats the other cost criterions such as expected average-cost criterion and expected total cost criterion when the time horizon is finite.

2. System Description

In this section, we present the notations and conditions which are necessary to formulate our problem as a Markov Decision Process problem. To begin with, we list the basic assumptions which are imposed upon system.

Assumptions

(1) The system consists of two series units.

(2) Each replacement takes one unit of time.

(3) If one unit or both is failed or replaced, the system is inoperative.

(4) While one unit is being replaced, the other unit does not deteriorate during this period.

State Space

As noted in the previous section we must consider the states of both unit 1 and unit 2. Thus the state space is defined as follows:

$S=\{(i,r);\ i$ is the state of unit 1 and r is the state of unit 2, $i=0,1,\cdots,\ r=0,1,\cdots\}$,

where state 0 is a "new" state, and other states $1,2,\ldots$ denote some degree of deterioration in ascending order.

Decision Space

Since there are two actions for each unit, that is, to replace or do nothing, the decision space consists of four actions as follows:

$A=\{a_0, a_1, a_2, a_3\}$, where

action a_0 is to leave the two units in service (no action), action a_1 is to replace unit 1 only, action a_2 is to replace unit 2 only, and action a_3 is to replace both unit 1 and unit 2.

Cost Structure

1) Each time the sytem is in state (i, r) and action a_0 (no action) is taken, an expected operating cost C_{ir} is incurred.
2) When unit 1 is replaced and unit 2 is not replaced (action a_1), a replacement cost R_1 is incurred.
3) When unit 2 is replaced and unit 1 is not replaced (action a_2), a replacement cost R_2 is incurred.
4) When both unit 1 and 2 are replaced (action a_3), a replacement cost R_{12} is incurred.

Transition Probability

1) If action a_0 is chosen at time t, then there are known transition probabilities p_{ij}, $i, j=0, 1, \dots$, and q_{rs}, $r, s=0, 1, \dots$ which satisfy

   $P\{X_{t+1}=j,\ Y_{t+1}=s \mid X_t=i,\ Y_t=r,\ \Delta_t=a_0\}=p_{ij}q_{rs}$ for all i, r, j and, s where

   X_t denotes state of unit 1 in use at time t

   Y_t denotes state of unit 2 in use at time t

   and

   Δ_t is the action chosen at time t.
2) If action a_1 is chosen at time t, then the transition probabilities for unit 1 and unit 2 are as follows:

   $$P\{X_{t+1}=j,\ Y_{t+1}=s \mid X_t=i,\ Y_t=r,\ \Delta_t=a_1\}=\begin{cases}1 & \text{if } j=0 \text{ and } s=r\\ 0 & \text{otherwise}\end{cases}$$

   This implies that while one unit is being replaced, the other unit does not deteriorate,
3) Similarly, if action a_3 is chosen at time t, then the transition probabilities for unit 1 and unit 2 are as follows:

   $$P\{X_{t+1}=j,\ Y_{t+1}=s \mid X_t=i,\ Y_t=r,\ \Delta_t=a_2\}=\begin{cases}1 & \text{if } j=i \text{ and } s=0\\ 0 & \text{otherwise}\end{cases}$$
4) If action a_3 is chosen at time t, then the transition probabilities for unit 1 and unit 2 are as follows:

   $$P\{X_{t+1}=j,\ Y_{t+1}=s \mid X_t=i,\ Y_t=r,\ \Delta_t=a_3\}=\begin{cases}1 & \text{if } j=0 \text{ and } s=0\\ 0 & \text{otherwise}\end{cases}$$

Conditions

We impose the following conditions on the costs and transition probabilities.

Condition 1 : $\{C_{ir}\}$, $i=0, 1, \dots$, $r=0, 1, \dots$, is a non-decreasing bounded sequence.

Condition 2 : $\max \}R_1, R_2\} < R_{12} < R_1 + R_2$.

Condition 3 : For each $k=0, 1, 2, \dots$, the function $R_k(i)=\sum_{j=k}^{\infty} p_{ij}$ is a non-decreasing function of $i=0,1,2,\ldots$, and the function $S_k(r)=\sum_{s=k}^{\infty} q_{rs}$ is a non-decreasing funtion of r for $r=0,1,2,\ldots$

Condition 1 says that the operating cost is non-decreasing function of the state. Condition 2 says that it costs less to replace two units concurrently than to replace them at different times. Condition 3 says that the conditional probability of a transiton into any block of states $\{k,k+1,\ldots\}$, given that action a_0 is chosen, is a non-decreasing function of the present state $i(r)$ for unit 1 (unit 2).

The following isolated result is listed for later use. For the proof, see Derman (2).

Lemma: Condition 3 implies that for every non-decreasing bounded sequence $\{h(j)\}\sum_{i=0}^{\infty}$ the funct ion $f(i)=\sum_{j=0}^{\infty} p_{ij}h(j)$ and $g(r)=\sum_{s=0}^{\infty} q_{rs}\,h(s)$ are also non-decreasing for $i=0$, $1,\ldots,$ $r=0,1,\ldots,$ respectively.

3. Discounted Cost Problem

The aim of this section is to find the policy R* which minimizes the total discounted cost incurred to the system over the infinite time horizon. In other words, we are going to find out the policy R* which satisfies

$$\Psi(i,r,R^*)=\min\ \Psi(i,r,R)$$

for any given initial state (i,r), where $\Psi(i,r,R)$ is defined as

$$\Psi(i,r,R)=\sum_{t=0}^{\infty}\alpha^t\ E_R[C(X_t,Y_t)\,|\,(X_o,Y_o)=(i,r)]$$

(Note: $\alpha(0\le\alpha<1)$ is a discount factor.)

Note that the policy R* is said to be an optimal α-discounted policy, or for short, a α-optimal policy. From here on, we shall use the simple notation $V(i,r)$ which is defined as $V(i,r)=\Psi(i,r,R^*)$, for ease of exposition.

In order to apply the famous Blackwell's functional equation to our problem,i t is necessary to show that $V(i,r)$ is finite for any initial sate (i,r). For that, let's check the values of the cost $C(X_t,Y_t)$ incurred to the system at time t at various cases.

$$C(X_t,Y_t)=\begin{cases} C_{ir} & if\ \Delta_t=a_0 \\ R_1 & if\ \Delta_t=a_1 \\ R_2 & if\ \Delta_t=a_2 \\ R_{12} & if\ \Delta_t=a_3 \end{cases}$$

(Δ_t is the action chosen at time t)

This, together with condition 1, ensures that cost incurred for one unit time is bounded and thus $V(i,r)$ is finite for any given state (i,r). We thus can apply Blackwell's functional equation to our problem. The reader is advised to refer to Ross(3) and Derman(2) for detailed reasons.

Now the functional equation of our problem is

$$V(i,r)=\min\{C_{ir}+\alpha\sum_j\sum_s p_{ij}\ q_{rs}\ V(j,s):\ R_1+\alpha V(o,r);\ \ R_2+\alpha V(i,o);\ \ R_{12}+\alpha V(o,o)\}\quad(1)$$

for $i=0,1,\ldots r=0,1,\ldots$ and any policy which prescribes action a_0 in state (i,r) when the first term is the minimum, action a_1 when the second term is the minimum, action a_2 when the third term is the minimum and action a_3 when the fourth term,s is the minimum, is the α-optimal policy.

An explicit solution of the funcitional equations for $V(i,r)$ is near impossible. However, an interaction technique for approaching functional equation is available. *Let* $V_o(i,r)=0$ for all i and r, and define successive approximation by

$$V_{k+1}(i,r)=\min\{C_{ir}+\alpha\sum_j\sum_s p_{ij}\,q_{rs}\,V_k(j,s):R_1+\alpha V_k(o,r);R_2+\alpha V_k(i,o);R_{12}+\alpha V_k(o,o)\}\cdots(2)$$

for all $i=0,1,\ldots,$ and $r=0,1,\ldots$

Intuitively, $V_k(i,r)$ is the cost if we follow optimal policy R^* for k periods and incur a terminal cost of zero, given we start in state (i,r).

The next three lemmas are necessary as intermediate steps for our main results.

Lemma 1 : Given $o\leq\alpha<1$, it follows that

$V_{k+1}(i,r)\geq V_k(i,r)$ for all i,r,k, and

$\lim_{k\to\infty}V_k(i,r)=V(i,r)$ for all i and r.

Proof

From equation (2), we have

$$V_1(i,r)=\min\}C_{ir}+\alpha\sum_j\sum_s p_{ij}\,q_{rs}\,V_o(j,s);\ R_1+\alpha V_o(o,r);\ R_2+\alpha V_o(i,o);\ R_{12}+\alpha V_o(o,o)\}$$
$$=\min\{C_{ir};\ R_1;R_2;\ R_{12}\}$$
$$\geq V_o(i,r)$$

Suppose $V_k(i,r)\geq V_{k-1}(i,r)$ for all i and r. Then

$$V_k{}^+{}_1(i,r)=\min\{C_{ir}+\alpha\sum_j\sum_s p_{ij}\,q_{rs}\,V_k(j,s);R_1+\alpha V_k(o,r);\ R_2+\alpha V_k(i,o);\ R_{12}+\alpha V_k(o,o)\}$$
$$\geq\min\{C_{ir}+\alpha\sum_j\sum_s P_{ij}\,q_{rs}\,V_{k-1}(j,s);\ R_1+\alpha V_{k-1}(o,r);\ R_2+\alpha V_{k-1}(i,o);\ R_{12}+\alpha V_{k-1}$$
$$V_k(i,r).=(o,o)\}$$

Thus $V_{k+1}(i,r)\geq V_k(i`r)$ for all $i,r,$ and k. Since the sequence $\{V_k(i,r)\Big|_{k=0}^{\infty}$ is bounded and non-decreasing, it converges. Suppose the sequence $\Big\{V_k(i,r)\Big\}_{k=0}^{\infty}$ converges to $V'(i,r)$. Then for any given $\varepsilon>0$, there exists a positive integer n_0 such that

$\max_{(i,r)}|V'(i,r)-V_n(i,r)|<\infty$ for all $n>n_0$.

Then for any $n>n_0$, we have

$$V_{n+1}(i,r)=\min\}C_{ir}+\alpha\sum_j\sum_s p_{ij}\,q_{rs}\,V_n(j,s);\ R_1+\alpha V_n(o,r);\ R_2+\alpha V_n(i,o);\ R_{12}+\alpha V_n(o,o)\}$$
$$\geq\min\}C_{ir}+\alpha\sum_j\sum_s p_{ij}\,q_{rs}\,[V'(j,s)\text{-}\varepsilon];\ R_1+\alpha[V'(i,o)\text{-}\varepsilon];\ R_2+\alpha[V'(i,o)\text{-}\varepsilon;\ R_{12}+\alpha\cdot[V'(o,o)\text{-}\varepsilon]$$
$$=\min\{C_{ir}+\alpha\sum_j\sum_s p_{ij}\,q_{rs}\,V'(j,s)\text{-}\alpha\varepsilon;\ R_1+\alpha V'(o,r)\text{-}\alpha\varepsilon;\ R_2\ +\alpha V'(i,o)\text{-}\alpha\varepsilon;\ R_{12}+\alpha V'(o,o)\text{-}\alpha\varepsilon\}$$
$$=\min\{C_{ir}+\alpha\sum_j\sum_s p_{ij}\,q_{rs}\,V'(j,s);\ R_1+\alpha V'(o,r);\ R_2+\alpha V'(i,o);R_{12}+\alpha V'(oo,)\}\text{-}\alpha\varepsilon$$

Let $V'(i,r)-V_{n+1}(i,r)=\delta<\varepsilon$, then

$$V'(i,\ r)\geq\min\}C_{ir}+\alpha\sum_j\sum_s p_{ij}\,q_{rs}\,V'(j,s);\ R_1+\alpha V'(o,r);R_2+\alpha V'(i,o);R_{12}+\alpha V'(o,o)\}\text{-}\alpha\varepsilon+\delta$$

Since ε is arbitrary, we have

$$V'(i, r) = \min\}C_{ir} + \alpha\sum_j\sum_s p_{ij}\ q_{rs} V'(j,s);\ R_1 + \alpha V'(or);\ R_2 + \alpha V'(i,o);\ R_{12} + \alpha V'(o,o)\}$$

Thus $V'(i,r)$ satisfies the functional equation (1).

By the uniqueness of equation (1) (see Ross (3)), $V'(i,r) = V(i,r)$ for all i and r; hence the proof is completed.

Lemma 2 : Under conditions (1), (2), and (3), $V_k(i,r)$ is non-decreasing for $i=0,1,\ldots$and $r=0,1,$ for each k.

Proof;

Since $V_o(i,r)=0$ for all i and r, $V_o(i,r)$ is non-decreasing for all i and r. Now suppose that $V_k(i,r)$ is non-decreasing for all i and r. Then

$$V_{k+1}(i,r) = \min\}C_{ir} + \alpha\sum_j\sum_s p_{ij}\ q_{rs}\ V_k(j,s);\ R_1 + \alpha V_k(o,r);\ R_2 + \alpha V_k(i,o);\ R_{12} + \alpha V_k(o,o)\}$$

$$\geq \min\{C_{i+1,r} + \alpha\sum_j\sum_s p_{i+1,j}q_{rs}\ V_k(j,s);\ R_1 + \alpha V_k(o,r);\ R_2 + \alpha V_k(i+1,0);\ R_{12} + \alpha V_k(o,o)\}$$

$$= V_{k+1}(i+1,r)$$

and

$$V_{k+1}(i,r) \geq \min\{C_{i,r+1} + \alpha\sum_j\sum_s p_{ij}\ q_{r+1,s}\ V_k(j,s);\ R_1 + \alpha V_k(o,r+1);\ R_2 + \alpha V_k(i,o);\ R_{12} + \alpha V_k(o,o)\} = V_{k+1}(i,r+1)$$

Thus $V(i,r)$ is non(decreasing for $i=0,1,$ $r=0,1,\ldots,$ and for all k.

Lemma 3 : Under conditions (1), (2), and (3) $V(i,r)$ is non-decreasing for $i=0,1,\ldots,$ and $r=0,1,\ldots.$

Proof : As $k\to\infty$, Lemma 1 and 2 imply that $V(i,r)$ is non-decreasing forall i and r.

Now define two functions F_r and G_i. F_r is the cost of replacing both units minus the cost of replacing unit 1 only and G_i is the cost of replacing both units minus the cost of replacing unit 2 only, respectively in state (i,r). From (1), we have

$F_r = R_{12} + \alpha V(o,o) - [R_1 + \alpha V(o,r)]$ and

$G_i = R_{12} + \alpha V(o,o) - [R_2 + \alpha V(i,o)]$.

By the montonicity of $V(i,r)$, F_r and G_i are non-decreasing in r and i, respectively. Hence there exists i* and r* such that

$$i^* = \min\{i;\ R_2 + \alpha V(i,o) > R_{12} + \alpha V(o,o)\}\ .$$

and

$$r^* = \min\{r:\ R_1 + \alpha V(o,r) > R_{12} + \alpha V(o,o)\}.$$

Next, wo consider two functions P_{ir} and Q_r as follows;

$$P_{ir} = \min\}C_{ir} + \sum_j\sum_s p_{ij}q_{rs}\ V(j,s);\ R_2 + \alpha V(i,o)\}$$

and

$$Q_r = \min\}R_1 + \alpha V(o,r);\ R_{12} + \alpha V(o,o)\}$$

Then we can write equation (1) as follows

$$V(i,r) = \min\{P_{ir};\ Q_r\} \tag{3}$$

In equation (3), when $V(i,r)=P_{ir}$, the optimal policy for unit 1 is no action and when $V(i,r)=Q_r$, the optimal policy for unit 1 is to replace. In either case, however, unit 2 may be replaced or not according to the state of unit 2.

Simiarly, we consider two functions R_{ir} and such that

$$R_{ir}=\min\{C_{ir}+\alpha\sum_j\sum_s p_{ij}\ q_{rs}\ V(j,s);\ R_1+\alpha V(o,r)\}$$

and

$$S_i=\min\{R_2+\alpha V(i,o);\ R_{12}+\alpha V(o,o)\}$$

Then we can write equation (1) as follows:

$$V(i,r)=\min\{R_{ir};S_i\} \qquad (4)$$

In equation (4), when $V(i,r)=R_{ir}$, the optimal policy for unit 2 is no action and when $V(i,r)=S_i$ the optimal policy for unit 2 is to replace. Therefore, we can obtain optimal policies for unit 1 and unit 2 from equations (3) and (4), respectively.

We are now in a position to present the following main Theorem in this paper.

Theorem 4 : Under conditions (1), (2), and (3) there exist control limits $i^*(r)$ for $r=0, 1, \ldots,$ and $r^*(i)$ for $i=0,1,\ldots$ such that

(a) In observed state (i,r), optimal policy for unit 1 is to replace if $i\geq i^*(r)$ and no action otherwise.

(b) In observed state (i,r), optimal policy for unit 2 is to replace if $r\geq r^*(i)$ and no action otherwise.

(c) $i^*(r)=\min\{i : P_{ir}>Q_i\}$ for each $r=0,1,\ldots,$ and $r^*(i)=\min\{r : R_{ir}>S_i\}$ for each $i=0,1,\ldots$

Proof:

We first prove the part (a) of the Theorem. Since P_{ir} is non-decreasing for all i and Q_r is independent of i, $J_{ir}=P_{ir}-Q_r$ is non-decreasing for all i. Then for all states (i,r) when $i\geq i^*(r)$, $P_{ir}>Q_r$. Hence replacing unit 1 minimizes the cost objective.

Simlarly $K_{ir}=R_{ir}-S_i$ is non-decreasing for all i and for all states (i,r) where $r\geq r^*(i)$, $R_{ir}>S_i$. Hence replacing unit 2 minimizes the cost objective. The above structure of optimal policy follows.

Corollary 4 : Under conditions (1), (2), and (3) the control limits $i^*(r)$ and $r^*(i)$ satisfy the following properties.

a) $i^*(r)\geq i^*$ for all $r\geq r^*$ and $i^*(r)$ is non-increasing in $r\geq r^*$.

b) $r^*(i)\geq r^*$ for all $i\geq i^*$ and $r^*(i)$ is non-increasing in $i\geq i^*$.

Proof:

a) Suppose $i^*(r)<i^*$ for some $r\geq r^*$, then there exists $i<i^*$ such that $P_{ir}>Q_r$, that is, $\min\{C_{ir}+\alpha\sum_j\sum_s p_{ij}\ q_{rs}\ V(j,s);\ R_2+\alpha V(i,o)\}>\min\{R_2+\alpha V(o,r);\ V_{12}+\alpha V(o,o)\}$. Thus

$$R_2+\alpha V(i,o)>\min\{R_1+\alpha V(o,r);\ R_{12}+\alpha V(o,o)\}$$

Since $i<i^*$ and $r\geq r^*$, we have, $R_2+\alpha V(i,o)<R_{12}+\alpha V(o,o)<R_1+\alpha V(o,r)$. This is a contradiction. Hence $i^*(r)\geq i^*$ for all $r\geq r^*$.

Next we show the monotonicity of $i^*(r)$ for all $r\geq r^*$.

For any $i\geq i^*(r)$ and $r\geq r^*$, we have

$$R_{12}+\alpha V(o,o)<\min\{R_1+\alpha V(o,r);\ R_2+\alpha V(i,o)\}\text{ and } P_{ir}>Q_r.$$

Since $P_{ir}=\min\{C_{ir}+\alpha\sum_j\sum_s p_{ij}q_{rs}\ V(j,s);\ R_2+\alpha V(i,o)\}$ and $Q_r=\min\{R_1+\alpha V(o,r);R_{12}+\alpha V(o,o)\}=R_{12}+\alpha V(o,o)$, $C_{ir}+\alpha\sum_j\sum_s p_{ij}\ q_{rs}\ V(j,s)>R_{12}+\alpha V(o,o)$.

By the montonicity of $C_{ir}\ +\alpha\sum_j\sum_s p_{ij}q_{rs}V(j,s)$ and $R_1+\alpha V(o,r)$,

$C_{ir+1}\ +\alpha\sum_j\sum_s p_{ij}q_{r+1,s}\ V(j,s)>\ R_{12}+\alpha V(o,o)$ and $R_1+\alpha V(o,r+1)>R_{12}+\alpha V(o,o)$.

Therefore

$\min\{C_{i,r+1}\ +\alpha\sum_j\sum_s p_{ij}\ q_{r+1,s}V(j,s);\ R_2+\alpha V(i,o)\}>\min\{R_1+\alpha V(o,r+1);\ R_{12}+\alpha V(o,o)\}$

and hence $i^*(r)\geq i^*(r+1)$. Thus $i^*(r)$ is non-increasing in $r\geq r^*$.

b) Similarly, we can also show that $r^*(i)\geq r^*$ for all $i\geq i^*$ and $r^*(i)$ is non-increasing in $i\geq i^*$.

Notes:

(1) The example presented in this section and many other examples say that in most cases $r^*(i)$and $i^*(r)$ are non-increasing for each $i=0,1,\ldots,$ and $r=0,1,\ldots,$ respectively. Especially, it is true for almost all pure deterioration processes.

(2) By Theorem 4, corollary 4, and Note (1), we can show that the structure of optimal policies are as shown in Fig.1. However, in most cases, structure of optimal policy is simpler than the structure shown in Fig.1.

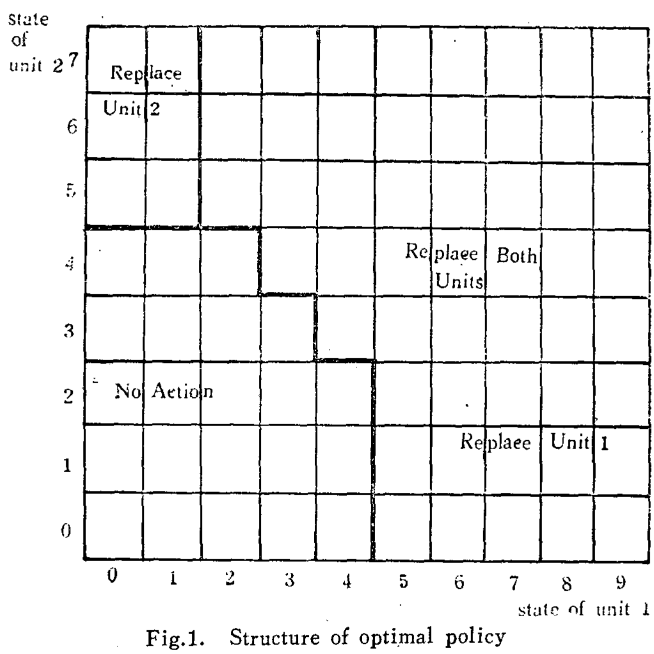

Fig.1. Structure of optimal policy

Numerical Example

Consider a series system of two units, i. e., unit 1 and unit 2, whose performances at any time $t=0,1,\ldots$ can be characterized by one of 10 states for unit 1 and one of 8 states for unit 2. The cost structure and deterioration processes of each unit are as below and discounted factor is set to $\alpha=0.9$. The structure of optimal policy is shown in Fig.2.

a. $C=(C_{ir})=\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 15 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 15 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 15 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 15 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 15 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 15 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 15 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 \end{pmatrix}$

b. $R_1=20,\ R_2=20,\ R_{12}=30$

c. $P=(p_{ir})=\begin{pmatrix} .6 & .2 & .1 & .1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .7 & .2 & .1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .6 & .2 & .1 & .1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .8 & .1 & .1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & .3 & .1 & .1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .6 & .1 & .1 & .1 & .1 \\ 0 & 0 & 0 & 0 & 0 & 0 & .7 & .1 & .1 & .1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .8 & .1 & .1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .9 & .1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \end{pmatrix}$

$Q=(q_{rs})=\begin{pmatrix} .7 & .2 & .1 & 0 & 0 & 0 & 0 & 0 \\ 0 & .8 & .1 & .1 & 0 & 0 & 0 & 0 \\ 0 & 0 & .6 & .2 & .1 & .1 & 0 & 0 \\ 0 & 0 & 0 & .7 & .1 & .1 & .1 & 0 \\ 0 & 0 & 0 & 0 & .8 & .1 & .1 & 0 \\ 0 & 0 & 0 & 0 & 0 & .9 & .1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .8 & .2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \end{pmatrix}$

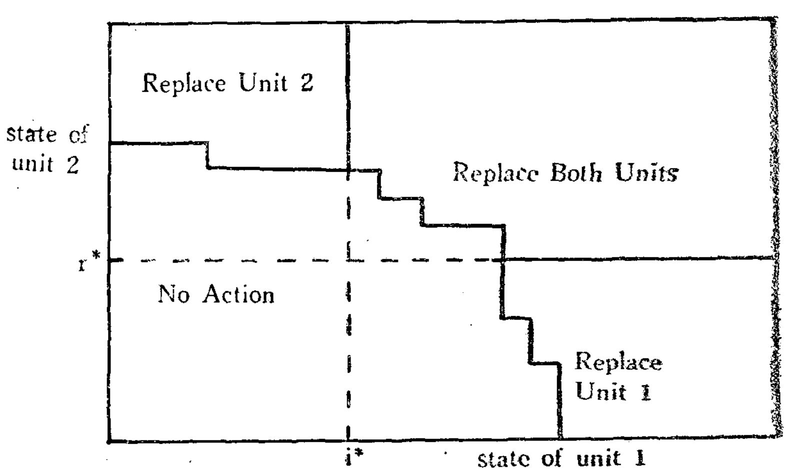

Fig. 2. Stucture of optimal policy of example 1.

4. Other Cost Criterion Problem

In this section we establish an optimal policy which minimizes the expected average-cost per unit time criterion and then determine an optimal polocy which minimizes the expected total cost when the time horizon (or planning horizon) is finite. In the finite time horizon case, we consider two cases; (1) time horizon is given, and (2) time horizon is not known with certainty but the pmf of time horizon is known.

4.1. Expected Average-Cost Problem

The structure of the optimal policy minimizing the expected average cost is similar to the structure of optimal policy minimizing the discounted-cost. Especially the two policies are exactly-same for all α near enough to 1. To do this we follow the approach suggested in Ross (3). For the discounted-cost problem, let's define $f_\alpha(i,r)=V(i,r)-V(o,o)$. Then the functional equation (1) can be rewritten as:

$$(1-\alpha)V(o,o)+f_\alpha(i,r)=\min\ \{C_{ir}+\alpha\sum_j\sum_s p_{ij}\ q_{rs}\ f_\alpha(j,s);R_1+\alpha f_\alpha(o,r);\ R_2+\alpha f_\alpha(i,o);\ R_{12}+f\}_\alpha(o,o)\} \quad (5)$$

when all the costs(C_{ir},R_1,R_2,R_{12}) are finite, we note that $V(i,r)-V(o,o)<\infty$ for all (i,r) and α. Under this condition, $f_\alpha(i,r)$ converges to a bounded function $f(i,r)$, and $(1-\alpha_n)$ $V(o,o)$ converges to a constant g for some sequence $\alpha_n\to 1$ (for the proof, see Ross(3)). In the limit, equation (5) becomes

$$g+f(i,r)=\min\{C_{ir}+\sum_j\sum_s p_{ij}\ q_{rs}\ f(j,s);R_1+f(o,r);\ R_2+f(i,o);\ R_{12}+f(o,o)\} \quad (6)$$

Then there exists an optimal policy π^* such that g is a minimum average cost and π^* is any policy which, for each i and r, prescribes action minimizing the right-hand side of equation(6).

Since $V(i,r)$ is non-decreasing for each i and r under conditions (1),(2), and (3), $f_\alpha(i,r)$ is also non-decreasing for every (i,r) and α. The convergence of $f_\alpha(i,r)$ to a bounded function $f(i,r)$ implies $f(i,r)$ is non-decreasing for all i and r. Define functions P_{ir}, Q_r, R_{ir}, and S_i as follows:

$$P_{ir}=\min\{C_{ir}+\sum_j\sum_s p_{ij}q_{rs}\ f(j,s);\ R_2+f(i,o)\}$$

$$Q_r=\min\{R_1+f(o,r);\ R_{12}+f(o,o)\}$$

$$R_{ir}=\min\{C_{ir}+\sum_j\sum_s p_{ij}q_{rs}\ f(j,s);\ R_1+f(o,r)\}$$

and

$$S_i=\min\{R_2+f(i,o);\ R_{12}+f(o,o)\}$$

Then we can easily obtain the structure of optimal policy. Let $i^*(r)=\min\{i : P_{ir}>Q_r\}$ and $r^*(i)=\min\{r : R_{ir}>S_i\}$ for each $i,r=0,1,\ldots$ The optimal policy for unit 1 is to replace if $i\geq i^*(r)$ and no action otherwise; and the optimal policy for unit 2 is to replace if $r\geq r^*(i)$ and no action otherwise.

4.2. Finite Horizon Expected Cost Problem

When the time horizon T is given, we can easily obtain following recursive formula that yields an optimal policy which minimizes the total cost of operating the system up to and including the time horizon T(see Derman(2)).

$$V_n^* \ (i,r)=\min\{C_{ir}+\sum_j\sum_s p_{ij}\ q_{rs}V^*_{n+1}(j,s);\ R_1+V^*_{n+1}(o,r);\ R_2+V^*_{n+1})i;o);\ R_{12}+V^*_{n+1}(o,o)\} \quad (7)$$

for $i=0,1,\ldots,$ $r=0,1,\ldots,$ and $n=0,1,\ldots,$ T; and $V_{T+1}(i,r)=0$ for all i and r.

Any policy which chooses action a_0 (no action) in state (i,r) at time n when the first term of the right-hand side of equation (7) is the minmum, action a_1 (replace unit 1 only) when the second term is the minimum, action a_2 (replace unit 2 only) when the third term is the minimum, and action a_3 (replace both units) when the fourth term is the minimum, is an optimal policy.

In the real stutations, it is not true that time horizon is fixed. Instead, the time horizon must be considered to be a r.v. A practical example of such a situation arises in replacing parts for a military aircraft which will become obsolete at some date in the future that is not known with certainty, but which can be described probabilistically. Here the time horizon is the time until obsolescence.

When the time horizon T is a finite r.v. with given pmf, f_i, $i=0,1,\ldots,$ m, the recursive formula that yields an optimal policy which minimizes the cost of operating the system up to and including the time horizon is

$$V^*_n(i,r)=\min\{C_{ir}+\bar{P}_n\sum_j\sum_s p_{ij}\ q_{rs}\ V^*_{n+1}(jj,s);\ R_1+\bar{P}_nV^*_{n+1}(o,r);\ R_2+\bar{P}_nV^*_{n+1}(i,o);\ R_{12}+P_nV^*_{n+1}(o,o)\} \quad (8)$$

for $i=0,1,\ldots$ $r=0,1,\ldots$ and $n=0.1,\ldots$ m; and

$V^*_{n+1}(i,r)=0$ for all i and r, where $\bar{P}_n=1-P_n$ and $P_n=f_n/(f_n+f_{n+1}+\cdots+f_m)$. Any policy which chooses the action that minimizes the right-hand side of equation (8) is an optimal policy. (For detail, see Chang(1)).

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