論 文

均一한 斷面을 가진 直線 핀에 對하여

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Analysis of A Straight Fin with Temperature-Dependent Thermal Conductivity

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要 約

핀(fin) 재료의 熱傳導係數는 溫度에 따라 直線的으로 變한다고 假定하여 一次元, 定常狀態의 解析을 하였다. 정밀해는 타원적분의 函數로 表示된다.

핀(fin) 효율은 핀끝의 온도의 함수로 주어졌다. 計算을 용이하게 하기 위하여 perturbation 解도 求하였다.

1. Introduction

Fins are widely used on heat exchangers to increase of heat transfer. Analysis of fins usually assumes one-dimensional steady-state heat conduction with constant thermal conductivity and heat transfer coefficient[1].

In this paper one-dimensional steady-state analysis is given for a straight fin with constant cross-sectional area and temperature dependent thermal conductivity. The thermal conductivity of the fin material is assumed to very linearly with temperature. An analytical solution is given in terms of elliptic integrals. Aziz and Enamul Hug [4] obtained a first order perturbation solution for the same problem. A second order perturbation solution is given for easy numerical computation.

2. Analysis

Temperature distribution in a straight fin satisfies following differential equation:

$$\frac{d}{dX}\left(kA\frac{dT}{dX}\right)-hP(T-T_\infty)=0 \qquad (1)$$

One-demensional steady-state is assumed. If the cross-sectional area of the fin A is constant, and the thermal conductivity of the fin material k varies linearly with temperature T, it can be expressed as

$$k=k_\infty\left(1+\beta\frac{T-T_\infty}{T_b-T_\infty}\right) \qquad (2)$$

Where k_∞ is thermal conductivity at surrounding temperature, T, and is a dimensionless constant. If we define

$$\theta=\frac{T-T_\infty}{T_b-T_\infty} \qquad (3)$$

$$x=X/L \qquad (4)$$

$$m=L\sqrt{hP/k_\infty A} \qquad (5)$$

the equation (1) can be made dimensionless

$$\frac{d}{dx}\left[(1+\beta\theta)\frac{d\theta}{dx}\right]-m^2\theta=0 \qquad (6)$$

Boundary conditions are

$$\theta=1 \text{ at } x=1 \qquad (7)$$

$$\frac{d\theta}{dx}=0 \text{ at } x=0 \qquad (8)$$

Epuation (7) assumes that the fin base temperature, T_b, is known, and equation (8) assumes that the fin tip is thermally insulated.

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Symbols are given in Nomenclature

2.1 Exact Solution

Equation (6) is a second order nonlinear differential equation, and exact solution is not yet given.

$$\text{Let } u=\frac{d\theta}{dx} \tag{9}$$

then equation (6) is transformed to a first order differential equation:

$$(1+\beta\theta)\ u\frac{du}{d\theta}+\beta u^2-m^2\theta=0 \tag{10}$$

Boundary condition (8) is changed to

$$u=0 \text{ at } x=0 \tag{11}$$

However, since x is missing in equation (10), boundary condition (11) is not useful, therefore, let

$$u=0 \text{ at } \theta=\theta_t \tag{12}$$

where θ_t is the value of θ at $x=0$, which is not known a priori.

Solution of equation (10) which satisfies the boundary condition (12) is easily obtained.

$$u^2=\frac{m^2\left\{\theta^2-\theta_t^2+\frac{2\beta}{3}(\theta^3-\theta_t^3)\right\}}{(1+\beta\theta)^2}$$

$$u=\frac{d\theta}{dx}=\frac{m}{1+\beta\theta}\sqrt{\theta^2-\theta_t^2+\frac{2\beta}{3}(\theta^3-\theta_t^3)} \tag{13}$$

Integrating equation (13), one obtains

$$mx=\int_{\theta_t}^{\theta}\frac{(1+\beta\theta)\,d\theta}{\sqrt{\theta^2-\theta_t^2+\frac{2\beta}{3}(\theta^3-\theta_t^3)}} \tag{14}$$

Applying boundary condition (7), θ_t is obtained from

$$m=\int_{\theta_t}^{1}\frac{(1+\beta\theta)\,d\theta}{\sqrt{\theta^2-\theta_t^2+\frac{2\beta}{3}(\theta^3-\theta_t^3)}} \tag{15}$$

when $\beta=0$, equation (14) and (15) are easily integrated to give widely known solution.

If $\beta\neq0$, the right-hand sides of equation (14) and (15) are integrated to give[2].

$$mx=\sqrt{\frac{6}{\beta\alpha_2}}\,(1+\beta\theta_t-\beta\alpha_1)\ F(\mathrm{a},\ \mathrm{b})-\alpha_1\sqrt{\frac{6\beta}{\alpha_2}}\,\pi(1;\mathrm{a},\ \mathrm{b}) \tag{16}$$

$$m=\sqrt{\frac{6}{\beta\alpha_2}}\,(1+\beta\theta_t-\beta\alpha_1)\ F(\mathrm{a}_b,\ \mathrm{b})-\alpha_1\sqrt{\frac{6\beta}{\alpha_2}}\,\pi(1;\ \mathrm{a}_b,\mathrm{b}) \tag{17}$$

where

$$\alpha_1=\frac{3}{4\beta}(1+2\beta\theta_t)-\sqrt{\left(1+\frac{2}{3}\beta\theta_t\right)(1-2\beta\theta_t)} \tag{18}$$

$$\alpha_2=\frac{3}{4\beta}(1+2\beta\theta_t)+\sqrt{\left(1+\frac{2}{3}\beta\theta_t\right)(1-2\beta\theta_t)} \tag{19}$$

$$\mathrm{a}=\text{arc }\sin\sqrt{\frac{\theta-\theta_t}{\theta-\theta_t+\alpha_1}} \tag{20}$$

$$\mathrm{a}_b=\text{arc }\sin\sqrt{\frac{1-\theta_t}{1-\theta_t+\alpha_1}} \tag{21}$$

$$\mathrm{b}=\sqrt{\frac{\alpha_2-\alpha_1}{\alpha_1}} \tag{22}$$

$$F(\mathrm{a},\mathrm{b})=\int_0^{\sin a}\frac{dt}{(1-t^2)(1-b^2t^2)} \tag{23}$$

is elliptic integral of the first kind, and

$$\pi(n;\ \mathrm{a},\mathrm{b})=\int_0^{\sin a}\frac{dt}{(1-\mathrm{n}t^2)\sqrt{(1-t^2)(1-b^2t^2)}} \tag{24}$$

is elliptic integral of the third kind, which are tabulated in mathematical tables, e.g., in Reference 2. Equations (16) and (17) are valid when both α_1 and α_2 are positive, i.e., $-\frac{1}{2}\leq\beta\theta_t\leq\frac{1}{2}$. Efficiency of a fin is usually defined as [Ref. 3]

$$\eta=\frac{\text{actual heat transferred}}{\text{heat which would be transferred if entire fin area were at base temperature}}$$

In dimensionless variables defined in this paper, this becomes

$$\eta=\frac{(1+\beta)\left.\frac{d\theta}{dx}\right|_{x=1}}{m^2} \tag{25}$$

cr,

$$\eta=\frac{\sqrt{1-\theta_t^2+\frac{2}{3}\beta(1-\theta_t^3)}}{m} \tag{26}$$

2.2. Perturbation Solution

Equation (16) with equation (17) is mathematically exact; however, since the temperature distribution is given in an implicit function of elliptic integrals, it is not useful for numerical computation. Therefore, an approximate solution by perturbation method is now to be obtained.

When $|\beta|\ll1$, θ can be approximated by

$$\theta=\theta_0+\beta\theta_1+\beta^2\theta_2+\cdots\cdots \tag{27}$$

Then substituting equation (27) into (6), and collecting terms with same power of β, one obtains

$$\frac{d^2\theta_0}{dx^2}-m^2\theta_0=0 \tag{28}$$

$$\frac{d^2\theta_1}{dx^2}-m^2\theta_1=-\theta_0\frac{d^2\theta_0}{dx^2}-\left(\frac{d\theta_0}{dx}\right)^2 \qquad (29)$$

$$\frac{d^2\theta_2}{dx^2}-m^2\theta_2=-\theta_0\frac{d^2\theta_1}{dx^2}-2\frac{d\theta_0}{dx}$$

$$\frac{d\theta_1}{dx}-\theta_1\frac{d^2\theta_0}{dx^2} \qquad (30)$$

Boundary conditions are

$$\theta_0=1 \text{ at } x=1, \text{ and } \frac{d\theta_0}{dx}=0 \text{ at } x=0 \qquad (31)$$

$$\theta_1=0 \text{ at } x=1, \text{ and } \frac{d\theta_1}{dx}=0 \text{ at } x=0 \qquad (32)$$

$$\theta_2=0 \text{ at } x=1, \text{ and } \frac{d\theta_2}{dx}=0 \text{ at } x=0 \qquad (33)$$

Solution of equations (28)−(33) are

$$\theta_0=\frac{\cosh mx}{\cosh m} \qquad (34)$$

$$\theta_1=\frac{\cosh 2m \cosh mx-\cosh m \cosh 2mx}{3\cosh^3 m} \qquad (35)$$

$$\theta_2=\left(\frac{2\cosh^2 2m}{9\cosh^5 m}-\frac{m \sinh m}{12\cosh^4 m}-\frac{3\cosh 3m}{16\cosh^4 m}\right)\cosh mx+\frac{mx \sinh mx}{12\cosh^3 m}-\frac{2\cosh 2m\cdot\cosh 2mx}{9\cosh^4 m}+\frac{3\cosh 3mx}{16\cosh^3 m} \qquad (36)$$

Efficiency of fin defined in equation (25) becomes

$$\eta=\frac{(1+\beta)}{m^2}\left[\frac{d\theta_0}{dx}+\beta\frac{d\theta_1}{dx}+\beta^2\frac{d\theta_2}{dx}+\cdots\right]_{x=1} \qquad (37)$$

3. Discussion and Conclusion

Exact solution (eq. 16) is not practical to calculate temperature distribution. However. complete temperature distribution in a fin is not important, and only heat transfer from a fin is important, which can be computed if fin efficiency is known. Sometimes temperature at the fin tip is a measure of the fin efficiency.

For a given value of β, relationship between θ_t and m can be computed from equation (17), and then efficiency as a function of θ_t is obtained from equation (26).

Table 1 shows values of perturbation functions, θ_0, θ_1, and θ_2, for m=0.2, 1.0, and 1.5. It is seen that values of the first and the second perturbation functions are very small, which indicate that the approximate solution by the perturbation method gives accurate results.

Fig. 1 shows fin efficiency, η as a function of dimensionless parameter m. For $\beta>0$, fin efficiency would be underestimated and for $\beta<0$, η would be overestimated if thermal conductivity is assumed constant.

Table 1.

Perturbation Functions for m=0.2

x	θ_0	θ_1	θ_2
0.0	0.98033	0.00434	−0.01847
0.1	0.98052	0.00430	−0.01829
0.2	0.98111	0.00418	−0.01776
0.3	0.98209	0.00397	−0.01687
0.4	0.98347	0.00367	−0.01562

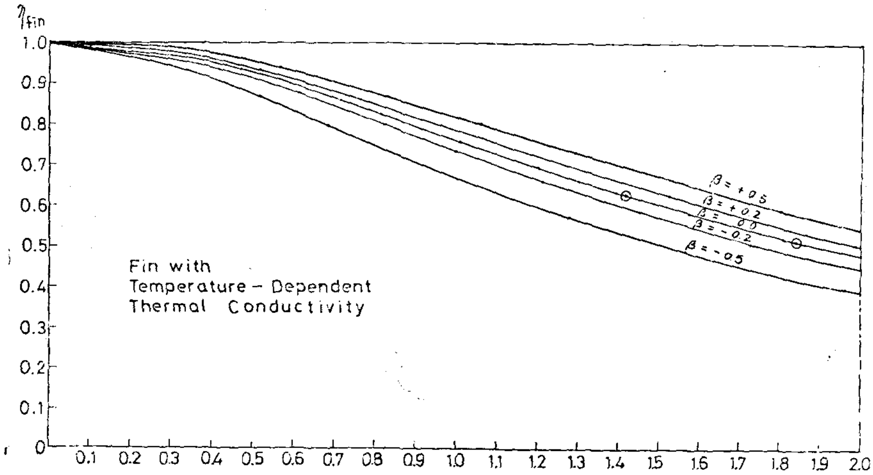

Fig. 1. Fin efficiency VS. dimensionless parameter m.

0.5	0.98523	0.00329	−0.01400
0.6	0.98739	0.00282	−0.01200
0.7	0.98995	0.00226	−0.00961
0.8	0.99290	0.00160	−0.00682
0.9	0.99625	0.00085	−0.00363
1.0	1.00000	0.00000	0.00000
$\frac{d\theta}{dx}\Big\vert_{x=1}$	0.03948	−0.03896	+0.02627

Perturbation Functions for m=1.0

x	θ_0	θ_1	θ_2
0.0	0.64805	0.00867	−0.08713
0.1	0.65130	0.00867	−0.08750
0.2	0.66106	0.00865	−0.08852
0.3	0.67744	0.00859	−0.08988
0.4	0.70059	0.00846	−0.09103
0.5	0.73076	0.00820	−0.09110
0.6	0.76825	0.00772	−0.08873
0.7	0.81342	0.00688	−0.08196
0.8	0.86673	0.00551	−0.06794
0.9	0.92872	0.00334	−0.04256
1.0	1.00000	0.00000	0.00000
$\frac{d\theta}{dx}\Big\vert_{x=1}$	0.76159	−0.61435	+0.71188

Perturbation Functions for m=1.5

x	θ_0	θ_1	θ_2
0.0	0.42510	0.19756	−0.02695
0.1	0.42989	0.19773	−0.03159
0.2	0.44437	0.19807	−0.03566
0.3	0.46887	0.19801	−0.04235
0.4	0.50394	0.19654	−0.05138
0.5	0.55036	0.19206	−0.06202
0.6	0.60920	0.18226	−0.07272
0.7	0.68177	0.16381	−0.08038
0.8	0.76970	0.13205	−0.07914
0.9	0.87500	0.08405	−0.05844
1.0	1.00000	0.00000	0.00000
$\frac{d\theta}{dx}\Big\vert_{x=1}$	1.35772	−0.98693	+0.86614

NOMENCLATURE

A: cross sectional area (m^2)

a: constant defined by eq. (20) (dimensionless)

b: constant defined by eq. (22) (dimensionless)

$F(a,b)$: elliptic integral of the first kind (dimensionless)

h: surface heat transfer coefficient (W/m^2 K)

k: thermal conductivity of fin material (W/m·K)

L: length of fin (m)

m: parameter defined by eq. (5) (dimensionless)

p: Perimeter of fin (m)

T: temperature (K)

u: nondimensionalized temperature gradient (dimensionless)

X: distance measured from fin tip (m)

x: nondimensionalized space coordinate (dimensionless)

Greek letters

d_1, α_2: constants defined by eq. (18) and (19) (dimensionless)

β: constant defined by eq. (2) (dimensionless)

η: efficiency of fin (dimensionless)

θ: nondimensionalized temperature (dimensionless)

$\theta_0, \theta_1, \theta_2$: perturbation functions defined by eq. (27) (dimensionless)

$(n;a,b)$: elliptic integral of the third kind (dimensionless)

Subscript

b: fin base condition

t: fin tip condition

∞: surrounding condition

REFERENCE

1. Jakob, M. Heat Transfer, Wiley, 1949, pp. 207-243.
2. Abramowitz, M., and I.A. Stegun, Handbook of Mathematical Functions, Dover Publication, 1965, pp. 589-628.
3. Holman, J.P., Heat transfer, 4th edition, Megrax-Hill, 1976, p.38
4. A. Aziz S.M. Enamul Hug, "Perturbation Solution for Convecting Fin with Variable Thermal Conductivity", J. Heat Tranfer, Vol. 97, pp. 300-301, May, 1975.