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INTEGRABILITY CONDITIONS OF STRUCTURES SATISFYING $f^k \pm f^r = 0$, $(k \geq 2r)$

By S.C. Rastogi and V.C. Gupta

1. Preliminaries

Let M^n be an n-dimensional differentiable manifold of class C^∞ equipped with a (1.1) tensor field $f(\neq 0)$ of class C^∞ satisfying

(1.1) $\qquad f^k \pm f^r = 0, \ (2 \text{ rank } f - \text{rank } f^{k-r}) = \dim M^n.$

Let l and m be the operators defined by

(1.2) $\qquad l = \mp f^{k-r}, \ m = I \pm f^{k-r}, \ l + m = I,$

where I is the identity operator, then these operators applied to the tangent space at a point of the manifold are complementary projection operators.

Let L and M be the complementary distributions corresponding to the projection operators l and m and the rank of f be p (a constant) then from (1.1) we obtain dim $L = (2p - n)$ and dim $M = (2n - 2p)$, where $n \leq 2p \leq 2n$. Such structures have been called by the authors [4] $f(k, \pm r)$-structures of rank p and the manifold M^n with these structures $f(k, \pm r)$-manifolds.

A tensor field f satisfying (1.1) and (1.2) also satisfies

(1.3) $\qquad f^r l = l f^r = f^r, \quad f^r m = m f^r = 0,$

and

(1.4) $\qquad f^{k-r} l = l f^{k-r} = \mp l, \quad f^{k-r} m = m f^{k-r} = 0.$

If $F \equiv f^{(k-r)/2}$ then $F(k, \pm r)$-structures of maximal rank are almost complex and almost product structures respectively and of minimal rank are almost tangent structures.

2. Nijenhuis tensor of $f(k, \pm r)$-structure

Let f be an $f(k, \pm r)$-structure of rank p, then the Nijenhuis tensor $N(X,Y)$ of f is given by [5] as follows:

(2.1) $\qquad N(X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y]$

Using (1.2) in (2.1) we obtain

(2.2) $N(lX, lY) = [\mp f^{k-r+1}X, \mp f^{k-r+1}Y] - f[\mp f^{k-r+1}X, lY] - f[lX, \mp f^{k-r+1}Y] + f^2[lX, lY]$;

(2.3) $N(lX, mY) = [\mp f^{k-r+1}X, fY \pm f^{k-r+1}Y] - f[\mp f^{k-r+1}X, mY] - f[lX, fY \pm f^{k-r+1}Y] + f^2[lX, mY]$;

(2.4) $N(mX, lY) = [fX \pm f^{k-r+1}X, \mp f^{k-r+1}Y] - f[fX \pm f^{k-r+1}X, lY] - f[mX, \mp f^{k-r+1}Y] + f^2[mX, lY]$;

and

(2.5) $N(mX, mY) = [fX \pm f^{k-r+1}X, fY \pm f^{k-r+1}Y] - f[fX \pm f^{k-r+1}X, mY] - f[mX, fY \pm f^{k-r+1}Y] + f^2[mX, mY]$.

Equations (2.2), (2.3), (2.4) and (2.5) in consequence of $l+m=I$, $lm=ml=0$ and (2.1) yield

(2.6) $N(X,Y) = N(lX, lY) + N(lX, mY) + N(mX, lY) + N(mX, mY)$.

If the distribution L is integrable $N(lX, lY)$ becomes the Nijenhuis tensor of $F_L \overset{\text{def}}{=} F/L$. Similarly if the distribution M is integrable $N(mX, mY)$ becomes the Nijenhuis tensor of $F_M \overset{\text{def}}{=} F/M$.

Let $\mathscr{L}_Y f$ be the Lie-derivative of the tensor field f with respect to a vector field Y, then we have [2]:

(2.7) $(\mathscr{L}_Y f)X = f[X, Y] - [fX, Y]$,

where $\mathscr{L}_Y f$ is a tensor field of the same type as f.

From (2.1) and (2.7) we obtain

(2.8) $N(lX, mY) = f(\mathscr{L}_{mY} f)lX - (\mathscr{L}_{fmY} f)lX$

and

(2.9) $N(mX, lY) = f(\mathscr{L}_{lY} f)mX - (\mathscr{L}_{flY} f)mX$.

3. Integrability conditions

THEOREM 3.1. *For any two vector fields X and Y the following hold:*

(i) *the distribution L is integrable if and only if $m \cdot N(lX, lY) = 0$;*

(ii) *the distribution M is integrable if and only if $l \cdot N(mX, mY) = 0$;*

(iii) *the distributions L and M are both integrable if and only if*

$$N(X,Y) = l \cdot N(lX, lY) + N(lX, mY) + N(mX, lY) + m \cdot N(mX, mY).$$

PROOF. We know that [2] for any two vector fields X and Y the distributions L and M are integrable if and only if $m \cdot [lX, lY] = 0$ and $l \cdot [mX, mY] = 0$

respectively. By virtrue of (1.2), (1.4) and (2.1) the first two parts are easily proved.

Since it is possible to write equation (2.6) as

$$N(X, Y) = l \cdot N(lX, lY) + m \cdot N(lX, lY) + N(lX, mY)$$
$$+ N(mX, lY) + l \cdot N(mX, mY) + m \cdot N(mX, mY),$$

therefore using the first and second part of the theorem we easily get the third part.

THEOREM 3.2. *If the distribution L is integrable, a necessary and sufficient condition for the almost complex structure(almost product structure) defined by F_L on each integral manifold of L to be integrable is that, for any two vector fields X and Y*

(3.1) $N(lX, lY) = 0.$

PROOF. Suppose that the distribution L is integrable then F induces on each integral manifold of L an almost complex structure(almost product structure). Since the induced structure is integrable if and only if its Nijenhuis tensor vanishes identically, therefore the result.

THEOREM 3.3. *If the distribution M is integrable, a necessary and sufficient condition for the almost tangent structure defined by F_M on each integral manifold of M to be integrable is that, for any two vector fields X and Y*

(3.2) $N(mX, mY) = 0.$

PROOF. The proof follows from the pattern of the proof of theorem (3.2).

DEFINITION 3.1. We say that $f(k, \pm r)$-structure is l-partially integrable if the distribution L is integrable and the almost complex structure(almost product structure) F_L induced from F on each integral manifold of L is also integrable.

DEFINITION 3.2. We say that $f(k, \pm r)$-structure is m-partially integrable if the distribution M is integrable and the almost tangent structure F_M induced from F on each integral manifold of M is also integrable.

DEFINITION 3.3. We say that $f(k, \pm r)$-structure is partially integrable if and only if it is both l-partially integrable and m-partially integrable.

THEOREM 3.4. *For any two vector fields X and Y, a necessary and sufficient condition for $f(k, \pm r)$-structure to be*

(i) *l-partially integrable is that* $N(lX, lY) = 0,$

(ii) *m-partially integrable is that* $N(mX, mY) = 0,$

(iii) *partially integrable is that*

$$N(X, Y)=N(lX, mY)+N(mX, lY).$$

PROOF. (i) The proof follows from theorems (3.1) (i) and (3.2).

(ii) The proof follows from theorems (3.2) (ii) and (3.3).

(iii) The proof follows from equations (2.6), (3.1) and (3.2).

THEOREM 3.5. *For any two vector fields X and Y,*

(i) *the tensor field $l(\mathscr{L}_{mY}f)l$ vanishes identically if and only if $N(lX, mY)=0$,*

(ii) *the tensor field $m(\mathscr{L}_{lY}f)m$ vanishes identically if and only if $N(mX, lY)=0$.*

PROOF. In consequence of (2.8), we have $N(lX, mY)=0$, if and only if

$$f(\mathscr{L}_{mY}f)lX=(\mathscr{L}_{fmY}f)lX.$$

Therefore if $N(lX, mY)=0$, we obtain

$$\begin{aligned} f^{k-r}(\mathscr{L}_{mY}f)lX &= f^{k-(r+1)}(\mathscr{L}_{fmY}f)lX \\ &= f^{k-(r+2)}(\mathscr{L}_{f^2mY}f)lX \\ &= \cdots\cdots\cdots \\ &= \cdots\cdots\cdots \\ &= f^{k-(r+k-r)}(\mathscr{L}_{f^{k-r}mY}f)lX \\ &= 0, \text{ in view of } (1.4). \end{aligned}$$

Thus by virtue of (1.2) the tensor field $l(\mathscr{L}_{mY}f)\,l$ vanishes identically, for any vector field Y.

(ii) The proof of this part is similar to that of (i).

4. Adapted coordinate system

When the distributions L and M are both integrable, we can choose a local coordinate system, such that all L are represented by putting $(2n-2p)$ local coordinates constant and all M are represented by putting the other $(2p-n)$ coordinates constant. We call such a coordinate system an adapted coordinate system.

We can suppose that in an adapted coordinate system the projection operators l and m have the components of the form

$$l=\begin{pmatrix} I_{2p-n} & 0 \\ 0 & 0 \end{pmatrix}, \quad m=\begin{pmatrix} 0 & 0 \\ 0 & I_{2n-2p} \end{pmatrix}, \tag{4.1}$$

respectively, where I_{2p-n} is a unit matrix of order$(2p-n)$ and I_{2n-2p} is of order $(2n-2p)$.

Since the distributions L and M are integrable, $fL \subset L$ and $fM \subset M$. Therefore the tensor f has the components of the form

$$f = \begin{pmatrix} f_{2p-n} & 0 \\ 0 & f_{2n-2p} \end{pmatrix} \tag{4.2}$$

in an adapted coordinate system. In (4.2) f_{2p-n} and f_{2n-2p} are square matrices of order $(2p-n) \times (2p-n)$ and $(2n-2p) \times (2n-2p)$ respectively.

Thus for any vector field mY on M, the Lie-derivative $\mathscr{L}_{mY} f$ has components of the form

$$\mathscr{L}_{mY} f = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \tag{4.3}$$

THEOREM 4.1. *For any two vector fields X and Y, in case of distributions L and M being integrable, a necessary and sufficient condition for the local components f_{2p-n} of $f(k, \pm r)$-strctures to be fuctions independent of the coordinates, which are constant along the integral manifolds of L in an adapted coordinate system is that $N(lX, mY) = 0$.*

PROOF. Let for any two vector fields X and $Y, N(lX, mY)$ be zero, then from theorem (3.5) (i), the tensor field $l(\mathscr{L}\ mY\ f)\ l$ vanishes identically, for any vector field Y. Hence $L_1 = 0$. This implies that the components f_{2p-n} of $f(k, \pm r)$-structure are independent of the coordinates which are constant along the integral manifolds of the distribution L in an adapted coordinate system.

Conversely, if the components f_{2p-n} of $f(k, \pm r)$-structure are independent of these coordinates, $L_1 = 0$. Thus the tensor field $l(\mathscr{L}_{my} f) l$ vanishes identically for any two vector field Y. Hence for any two vector fields X and Y, $N(lX, mY) = 0$, which proves the theorem.

THEOREM 4.2. *For any two vector fields X and Y, in case of distributions L and M being integrable, a necessary and sufficient condition for the local components f_{2n-2p} of $f(k, \pm r)$—structures to be functions independent of the coordinates, which are constant along the integral manifolds of M in an adapted coordinate system is that $N(mX, lY) = 0$.*

PROOF. The proof of this theorem is similar to that of theorem (4.1).

DEFINITION 4.1. We say that $f(k, \pm r)$-structure is l-integrable if

(i) $f(k, \pm r)$-structure is l-partially integrable,

(ii) the components f_{2p-n} of $f(k, \pm r)$-structures are independent of the coordinates which are constant along the integral manifolds of L in an adapted coordinate system;

(iii) the components f_{2n-2p} of $f(k, \pm r)$-structures are independent of the coordinates which are constant along the integral manifolds of M in an adapted coordinate system.

DEFINITION 4.2. We say that $f(k, \pm r)$-structure is m-integrable if

(i) $f(k, \pm r)$-structure is m-partially integrable,

(ii) the condition (ii) and (iii) of definition (4.1) are satisfied.

DEFININION 4.3. We say that $f(k, \pm r)$-structure is integrable if $f(k, \pm r)-$structure is partially integrable and the conditions (ii) and (iii) of definition (4.1) are satisfied.

THEOREM 4.3. *The necessary and sufficient condition for $f(k, \pm r)$-structure, in case of two vector fields X and Y to be*

(i) *l-integrable is that $N(X, Y)=N(mX, mY)$,*

(ii) *m-integrable is that $N(X, Y)=N(lX, lY)$,*

(iii) *integrable is that $N(X, Y)=0$.*

PPOOF. The proof of this theorem follows by virtue of theorems (3.4), (4.1), (4.2) and definitions (4.1), (4.2) and (4.3).

University of Nigeria,
Nsukka,
Nigeria.

Lucknow University,
Lucknow,
India.

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