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ON A GENERALIZED DOUBLE INTEGRAL TRANSFORM

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1. Introduction

The generalized M-function of two variables, occuring in this paper is defined by Mourya. D.P. [7] and is represented as follows:

(2.1)

$$M(x,y)\equiv M\left[\begin{matrix}\begin{pmatrix}m_1, & n_1\\ p_1-m_1, & q_1-n_1\end{pmatrix} \\ \begin{pmatrix}m_2, & n_2\\ p_2-m_2, & q_2-n_2\end{pmatrix} \\ \begin{pmatrix}m_3, & n_3\\ p_3-m_3, & q_3-n_3\end{pmatrix}\end{matrix}\left|\begin{matrix}\{(a_{p_1};\alpha_{p_1},\alpha_{p_1})\}; & \{b_{q_1};\beta_{q_1},\beta_{q_1})\} \\ \{(c_{p_2},\gamma_{p_2})\} & ;\ \{(d_{q_2}, \quad \delta_{q_2})\} \\ \{(e_{p_3},\epsilon_{p_3})\} & ;\ \{(f_{q_3}, \quad \rho_{q_3})\}\end{matrix}\right|\begin{matrix}x\\ \\ y\end{matrix}\right]$$

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} m(\xi,\eta)\ x^{\xi}y^{\eta}d\xi d\eta.$$

where $\{(a_{p_1};\ \alpha_{p_1},\ \alpha_{p_1})\}$ and $\{(c_p,\ \gamma_p)\}$ stand for the set of parameters $(a_1;\ \alpha_1,\ \alpha_1)$, $(a_2;\ \alpha_2,\ \alpha_2)$, $\cdots$, $(a_{p_1},\ \alpha_{p_1},\ \alpha_{p_1})$, and $(c_1,\ \gamma_1)$, $(c_2,\ \gamma_2)$, $\cdots$, $(c_p,\ \gamma_p)$ respectively and

$$(1.2)\quad m(\xi,\eta)=\frac{\prod_{j=1}^{m_1}\Gamma(1-a_j+\alpha_j\xi+\alpha_j\eta)\prod_{j=1}^{n_1}\Gamma(b_j-\beta_j\xi-\beta_j\eta)}{\prod_{j=m_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-\alpha_j\eta)\prod_{j=n_1+1}^{q_1}\Gamma(1-b_j+\beta_j\xi+\beta_j\eta)}$$

$$\times\frac{\prod_{j=1}^{m_2}\Gamma(1-c_j+\gamma_j\xi)\prod_{j=1}^{n_2}\Gamma(d_j-\delta_j\xi)}{\prod_{j=m_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)\prod_{j=n_2+1}^{q_2}\Gamma(1-d_j+\delta_j\xi)}$$

$$\times\frac{\prod_{j=1}^{m_3}\Gamma(1-e_j+\epsilon_j\eta)\prod_{j=1}^{n_3}\Gamma(f_j-\rho_j\eta)}{\prod_{j=m_3+1}^{p_3}\Gamma(e_j-\epsilon_j\eta)\prod_{j=n_3+1}^{q_3}\Gamma(1-f_j+\rho_j\eta)}.$$

where p_i, q_i, m_i and n_i $(i=1,2,3)$ are non-negative integers such that $0\leq m_i\leq p_i$,

$0 \leq n_i \leq q_i$. a_j, b_j, c_j, d_j, e_j and f_j are complex numbers and α_j, β_j, γ_j, δ_j, ϵ_j and ρ_j are positive real numbers.

No pole of $\Gamma(1-a_j+\alpha_j\xi+\alpha_j\eta)$, $\Gamma(1-c_j+\gamma_j\xi)$ and $\Gamma(1-e_j+\epsilon_j\eta)$ coincide with any pole of $\Gamma(b_j-\beta_j\xi-\beta_j\eta)$, $\Gamma(d_j-\delta_j\xi)$ and $\Gamma(f_j-\rho_j\eta)$ respectively.

L_1 and L_2 are suitable contours. x and y are not equal to zero and $x_\xi=\exp\{\xi(\log|x|+i\arg x)\}$;

$y^\eta=\exp\{\eta(\log|y|+i\arg y)\}$ in which $\log|x|$ and $\log|y|$ denote the natural logrithms of $|x|$ and $|y|$.

The integral on the right hand side of (1.1) is convergent under the following set of conditions.

(1.3) (i) $\mu_1 \equiv \Big[\sum_{j=1}^{m_1}\alpha_j+\sum_{j=1}^{n_1}\beta_j+\sum_{j=1}^{m_2}\gamma_j+\sum_{j=1}^{n_2}\delta_j-\sum_{j=m_1+1}^{p_1}\alpha_j$
$-\sum_{j=n_1+1}^{q_1}\beta_j-\sum_{j=m_2+1}^{p_2}\gamma_j-\sum_{j=n_2+1}^{q_2}\delta_j\Big]>0$

(ii) $\mu_2 \equiv \Big[\sum_{j=1}^{m_1}\alpha_j+\sum_{j=1}^{n_1}\beta_j+\sum_{j=1}^{m_3}\epsilon_j+\sum_{j=1}^{n_3}\rho_j-\sum_{j=m_1+1}^{p_1}\alpha_j$
$-\sum_{j=n_1+1}^{q_1}\beta_j-\sum_{j=m_3+1}^{p_3}\epsilon_j-\sum_{j=n_3+1}^{q_3}\rho_j\Big]>0$

(iii) $\mu_3 \equiv \Big[\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{q_2}\delta_j-\sum_{j=1}^{p_1}\alpha_j-\sum_{j=1}^{p_2}\gamma_j\Big]>0$

(iv) $\mu_4 \equiv \Big[\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{q_3}\rho_j-\sum_{j=1}^{p_1}\alpha_j-\sum_{j=1}^{p_3}\epsilon_j\Big]>0.$

(v) $|\arg x|<\frac{1}{2}\mu_1\pi$, $|\arg y|<\frac{1}{2}\mu_2\pi$.

we use the notation || for 'is replaced by'.

2. Definition

If $k(\lambda, x)$ is a function of variable x and parameter λ, defined on the interval (a, b), then the relation

$$T[f:\lambda]=\int_a^b k(\lambda, x)\, f(x)\, dx, \quad b>a \tag{2.1}$$

is called an integral transform of $f(x)$ with respect to the kernel $k(\lambda, x)$ over the interval (a, b). The domain of λ and the class of functions, to which $f(x)$ belongs are so prescribed that the integral (2.1) exists.

We introduce a double integral transform in the form

$$(2.2)\quad \phi[f:\lambda,\ \mu]\equiv\phi^{(m_1,n_1),\ (m_2,n_2),\ (m_3,n_3)}_{[p_1,q_1],\ [p_2,q_2],\ [p_3,q_3]}\left[f(x,y):\lambda,\mu\left|\begin{matrix}\{(a_j;\ \alpha_j,\ d_j)\}; & \{(b_j;\ \beta_j,\ \beta_j\}\\ \{(c_j,\ \gamma_j)\} & ;\ \{(d_j,\ \delta_j)\}\\ \{(e_j,\ \epsilon_j)\} & ;\ \{(f_j,\ \rho_j)\}\end{matrix}\right.\right]$$

$$=\int_0^\infty\int_0^\infty e^{-\frac{\lambda x}{2}-\frac{\mu y}{2}}M\left(\frac{\lambda^{2p}x^{2p}}{m^{2p}},\ \frac{\mu^{2q}y^{2q}}{n^{2q}}\right)f(x,y)\ dx\ dy$$

where the M-function is defined in section 1. The transform defined above exists if conditions in (1.3) with x, y replaced by λ, μ respectively are satisfied and

$$x^{\frac{2pd_i}{\delta_i}}y^{2qf_j/\rho_j}f(x,y)\in L(0,\delta),\ \delta>0$$

for $i=1,2,\cdots,n_2$. $j=1,2,\cdots,n_3$.

3. The Inversion Formula

We now establish the following theorem which provides us with a solution of the integral equation (2.2), solved for the unknown function $f(x,y)$ in terms of its image $\phi[f:\lambda,\ \mu]$

THEOREM *If* $K(x,y)=\dfrac{1}{(2\pi i)^2}\displaystyle\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma'-i\infty}^{\sigma'+i\infty}\frac{x^{-\xi}y^{-\eta}}{\phi(1-\xi,\ 1-\eta)}d\xi\ d\eta$

then

$$f(x,y)=\int_0^\infty\int_0^\infty K(\lambda x,\ \mu y)\,\phi\,[f:\ \lambda,\ \mu]\,d\lambda\ d\mu$$

where

$$\phi(\xi,\ \eta)=\frac{2}{\pi}^{2\xi+2\eta-2}M\left[\begin{matrix}\begin{bmatrix}m_1 & , & n_1\\ p_1-m_1, & & q_1-n_1\end{bmatrix}\\[2ex] \begin{pmatrix}m_2+2 & , & n_2\\ p_2-m_2, & & q_2-n_2\end{pmatrix}\\[4ex] \begin{pmatrix}m_3+2 & , & n_3\\ p_3-m_3, & & q_3-n_3\end{pmatrix}\end{matrix}\left|\begin{matrix}\{(a_{p_1};\ \alpha_{p_1},\ \alpha_{p_1})\};\ \{(b_{q_1};\ \beta_{q_1},\ \beta_{q_1})\}\\ \left(\frac{2-\xi}{2},\ p\right),\\ \left(\frac{1-\xi}{2},\ p\right),\quad \{(d_{q_2},\ \delta_{q_2})\}\\ \{(c_{p_2},\ \gamma_{p_2})\};\\ \left(\frac{2-\eta}{2},\ q\right),\\ \left(\frac{1-\eta}{2},\ q\right),\quad \{(f_{q_3},\ \rho_{q_3})\}\\ \{(e_{p_3},\ \varepsilon_{p_3})\};\end{matrix}\right|\begin{matrix}\dfrac{4^{2p}}{m^{2p}}\\[10ex] \dfrac{4^{2q}}{n^{2q}}\end{matrix}\right]$$

Provided $|K(x,y)|$ *exists,* $f(x,y)$ *is convergent and*

$$\lambda^{-\sigma}\mu^{-\sigma'}\phi[f:\lambda,\mu]\in L(0,\infty),$$

$$x^{\sigma-1}\ y^{\sigma'-1}\ f(x,y)\in L(0,\delta),\ \delta>0$$

$$2p\ Re\left(\frac{d_i}{\delta_i}\right)+1>\sigma \text{ for } i=1,2,\cdots,n_2,$$

$$2q\ Re\left(\frac{f_j}{\rho_j}\right)+1>\sigma' \text{ for } j=1,2,\cdots,n_3.$$

PROOF we have from (2.2)

$$\int_0^\infty\int_0^\infty \lambda^{-\xi}\mu^{-\eta}\phi[f:\lambda,\mu]\,d\lambda\,d\mu$$

$$=\int_0^\infty\int_0^\infty \lambda^{-\xi}\ \mu^{-\eta}\left\{\int_0^\infty\int_0^\infty e^{-\frac{\lambda x}{2}-\frac{\mu y}{2}}M\left(\frac{\lambda^{2p}x^{2p}}{m^{2p}},\ \frac{\mu^{2q}y^{2q}}{n^{2q}}\right)f(x,y)dx\ dy\right\}d\lambda\ d\mu.$$

Now interchanging the order of integrations which is easily justified by De la Vallee Poussin's theorem [3] under the conditions stated earlier, and making the substitution $\lambda x=u$, $\mu y=v$ in the inner integrals, we get

$$\int_0^\infty\int_0^\infty \lambda^{-\xi}\ \mu^{-\eta}\phi[f:\lambda,\mu]\,d\lambda\ d\mu=\int_0^\infty\int_0^\infty x^{\xi-1}\ y^{\eta-1}\ f(x,y)\ I\ dx\ dy$$

$$\text{where } I=\int_0^\infty\int_0^\infty e^{-\frac{u}{2}-\frac{v}{2}}M\left(\frac{u^{2p}}{m^{2p}},\ \frac{v^{2q}}{n^{2q}}\right)\ u^{-\xi}v^{-\eta}du\ dv$$

Substituting for M-function in terms of double Mellin-Barne's type contour integral, interchanging the order of integrations, evaluating the inner integral and again interpreting the result in terms of M-function we get $I=\phi(1-\xi,1-\eta)$. Hence we obtain the result:

$$\int_0^\infty\int_0^\infty \lambda^{-\xi}\ \mu^{-\eta}\phi[f:\lambda,\mu]\,d\lambda\ d\mu=\int_0^\infty\int_0^\infty x^{\xi-1}y^{\eta-1}f(x,y)dxdy\ \phi(1-\xi,1-\eta)$$

i.e.

$$\int_0^\infty\int_0^\infty x^{\xi-1}y^{\eta-1}f(x,y)dx\ dy=\int_0^\infty\int_0^\infty \frac{\lambda^{-\xi}\ \mu^{-\eta}}{\phi(1-\xi,\ 1-\eta)}\phi[f:\lambda,\mu]\,d\lambda\,d\mu=F(\xi,\eta) \text{ say}$$

Now by using Reed's theorem [12] we get

$$f(x,y)=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} x^{-\xi}y^{-\eta}\left\{\int_0^\infty\int_0^\infty \lambda^{-\xi}\ \mu^{-\eta}\frac{\phi[f:\lambda,\mu]}{\phi(1-\xi,\ 1-\eta)}\ d\lambda\ d\mu\right\}\ d\xi\ d\eta.$$

Interchanging the order of integrations the required result is obtained.

4. Particular Cases

On specialising the parameters suitably M-function can be reduced to H-function due to Munot and Kalla [8], P-function due to Pathak [10], Miejer's G-function of two variables due to Agrawal R.P. [1], S-function due to Sharma B.L. [13], hence from (2.2) we get double integral transforms involving these functions.

(i) If in (2.2) $m_1=n_1=0$, $p_1=A$, $q_1=B$, $m_2=q$, $n_2=r$, $p_2=C$, $q_2=D$, $m_3=k$, $n_3=1$, $p_3=E$, $q_3=F$, $c_j||\ 1-c_j$, $e_j=||1-e_j$, $p=q=1$, then (2.2) reduces to a double integral transform defined by Shrivastava H.M. [15]. Hence the transforms given by Bose S.K. [2], Sharma K.C. [14], Nigam H.N. [9], Verma R. U. [16] etc which are particular cases of the transform given by Shrivastava H.M. [15] will also be the particular cases of our transform (2.2)

(ii) Similarly on putting in (2.2) $p_1=m_1=n_1=q_1=0$, $p=q=1$, $m=n=4$, $\lambda=2p$, $\mu=2q$, $m_2=p_2=m_3=p_3=2$, $n_2=q_2=n_3=q_3=4$, $c_1=\frac{1}{2}$, $c_2=1$, $d_1=\frac{m-k}{2}$, $d_2=\frac{1+m-k}{2}$, $d_3=\frac{-m-k}{2}$, $d_4=\frac{1-m-k}{2}$, $\gamma_1=\gamma_2=\delta_1=\delta_2=\delta_3=\delta_4=\epsilon_1=\epsilon_2=\rho_1=\rho_2=\rho_3=\rho_4=1$ $e_1=\frac{1}{2}$, $e_2=1$, $f_1=\frac{m_1-k_1}{2}$, $f_2=\frac{1+m_1-k_1}{2}$, $f_3=\frac{-m_1-k_1}{2}$, $f_4=\frac{1-m_1-k_1}{2}$ and simplifying as in the previous case (2.2) gets reduced to the transform given by Mehra [6]

(iii) Also on putting in (2.2) $p=q=1$, $\lambda=2p$, $\mu=2q$, $m=2=n$, $p_1=m_1=q_1=n_1=0$, $m_2=p_2=m_3=p_3=2$, $n_2=q_2=n_3=q_3=4$, $c_1=\frac{1}{2}$, $c_2=1$, $d_1=\frac{2m+1}{4}$, $d_2=\frac{2m+3}{4}$, $d_3=\frac{-2m+1}{4}$, $d_4=\frac{1-m}{2}+\frac{1}{4}$, $e_2=\frac{1}{2}$, $e_2=1$, $f_1=\frac{m_1}{2}+\frac{1}{4}$ $f_2=\frac{m_1}{2}+\frac{3}{4}$ $f_3=\frac{-m_1}{2}+\frac{1}{4}$ $f_4=\frac{1-m_1}{2}+\frac{1}{4}$

and other parameters unity and simplifying as in the previous case, (2.2) reduces to the transform due to Rathie [11]. Here in addition we have to use the result $W_{0,\mu}(x)=\left(\frac{x}{\pi}\right)^{1/2}K_\mu(x/2)$ for simplification [5, p.432].

Certain interesting properties of this transform (2.2) have also been studied by the authors and have been communicated for publication.

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